Theorist®

VERSION 2.0

learning guide

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Acquired by Waterloo Maple Software, © 1993
Acquired by WebPrimitives, LLC © 1999
Acquired by MathMonkeys, LLC © 2003

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Theorist Learning Guide

First Edition, September 1994

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The following preparation instructions apply to all Macintosh System versions. Do not follow these instructions if you are using a computer with Microsoft Windows.

Disk Backup

Before you use the Theorist disks, you should make backup copies in case of errors or accidents which destroy the files on the disks. Your Macintosh manual explains how to copy a disk with the Finder, or you can copy the Theorist diskettes with a third party disk duplication program if you have one. After you have made copies, put the original Theorist diskettes in a safe place and do not use them; use only the backup copies.

Serial Number

Please take note of the serial number label on the Theorist application disks. This number is needed to install the program and is used when upgrading. Do not give your serial number to others. Write it in your manuals.

System Requirements

Theorist must be used on an Apple Macintosh Plus or later model Macintosh. (An older Macintosh upgraded to the equivalent of a Plus is satisfactory.) The minimum memory requirement is two megabytes (2Mb) total RAM. The Macintosh must be equipped with a hard disk, which should have at least 2Mb of free space for Theorist and its related files.

The Macintosh must be running System version 6.0.2 or a later version of Apple system software.

Symbol Font and Greek Letters

In order for you to use Greek letters and other symbols in your equations, you must have the Symbol font installed in your System. The Symbol font is supplied with all System versions released by Apple Computer.

The Symbol font can be found on one or more of the Apple Macintosh System software disks. Follow your Macintosh documentation to learn how to copy it to your System, if it is not already installed.

Choosing an Application Edition

The Theorist application on the disk labeled "Standard Application" is for use with the Macintosh Plus, SE, Portable, Classic, LC, Si, and other Mac models which have a 68000 central processor *or* models with other CPUs but no floating point processors. This application version, called the "standard version," *will* also run on other Macs, but it will not be as fast as the special "FPU version" (described below) designed for them.

The 68040 CPU used in the Centris and Quadra models has built-in FPU emulation.

The Theorist application on the disk labeled "FPU Application" is for use with the Macintosh II models, Quadras, SE/30, Classic II, certain PowerBooks, and other Mac models which have a 68020, 68030, or 68040 central processing unit *and* a 68881 or 68882 floating point unit. This application version, called the "FPU version," will *not* work on machines not so equipped.

The Theorist application on the disk labeled "PowerMac Application" is for use with PowerMacintosh models. Do *not* attempt to run this PowerMac edition on other Macs.

Installing Theorist on a Macintosh System

The following installation instructions apply to all Macintosh System versions. Do not follow these instructions if you are using a computer with Microsoft Windows.

Remove Old Versions

If you have an older version of Theorist, remove it and its notebook folders from your hard disk. (You can keep your own notebooks.)

Theorist Folder

With your Macintosh at the Finder (also called the desktop), create a new folder on your hard disk. Rename this empty folder to Theorist Folder.

Standard Application Disk

If you plan to use the standard edition (i.e., the non-FPU edition) of the Theorist application follow these instructions. If you plan to use the FPU edition or PowerMac edition, skip ahead to the heading "FPU Application Disk" or "PowerMac Application Disk"

Insert your backup copy of the disk labeled "Standard Application". The floppy disk icon appears on the desktop, with an open window. From this window, select the Theorist® file, the New Notebook file, and the TheoText.CAT file, and drag them into the Theorist folder on your hard disk. Drag the floppy disk icon to the trash to eject the disk. Skip ahead to the instructions under the "Notebooks Disk" heading.

FPU Application Disk

If you plan to use the FPU edition of the Theorist application follow these instructions. If you have already copied the standard version from the "Standard Application" disk, skip ahead to the instructions under the "Notebooks Disk" heading.

Insert your backup copy of the disk labeled "FPU Application". The floppy disk icon appears on the desktop, with an open window. From this window, select the Theorist®/FPU file, the New Notebook file, and the TheoText.CAT file, and drag them into the Theorist folder on your hard disk. Drag the floppy disk icon to the trash to eject the disk. Skip ahead to the instructions under the "Notebooks Disk" heading.

PowerMac Application Disk

If you plan to use the Theorist on a PowerMacintosh follow these instructions. If you have already copied an edition of Theorist from the "Standard Application" disk or the "FPU Application" disk, skip ahead to the instructions under the "Notebooks Disk" heading.

Insert your backup copy of the disk labeled "PowerMac Application". The floppy disk icon appears on the desktop, with an open window. From this window, select the Theorist®/PowerMac file, the New Notebook file, and the TheoText.CAT file, and drag them into the Theorist folder on your hard disk. Drag the floppy disk icon to the trash to eject the disk.

Notebooks Disk

Insert your backup copy of the disk labeled "Notebooks". The floppy disk icon appears on the desktop, with an open window. From this window, select the Mathematics folder, the Color Graphics folder, and the Graphics folder, and drag them into the Theorist folder on your hard disk. Drag the floppy disk icon to the trash to eject the disk.

Theorist Application Terminology

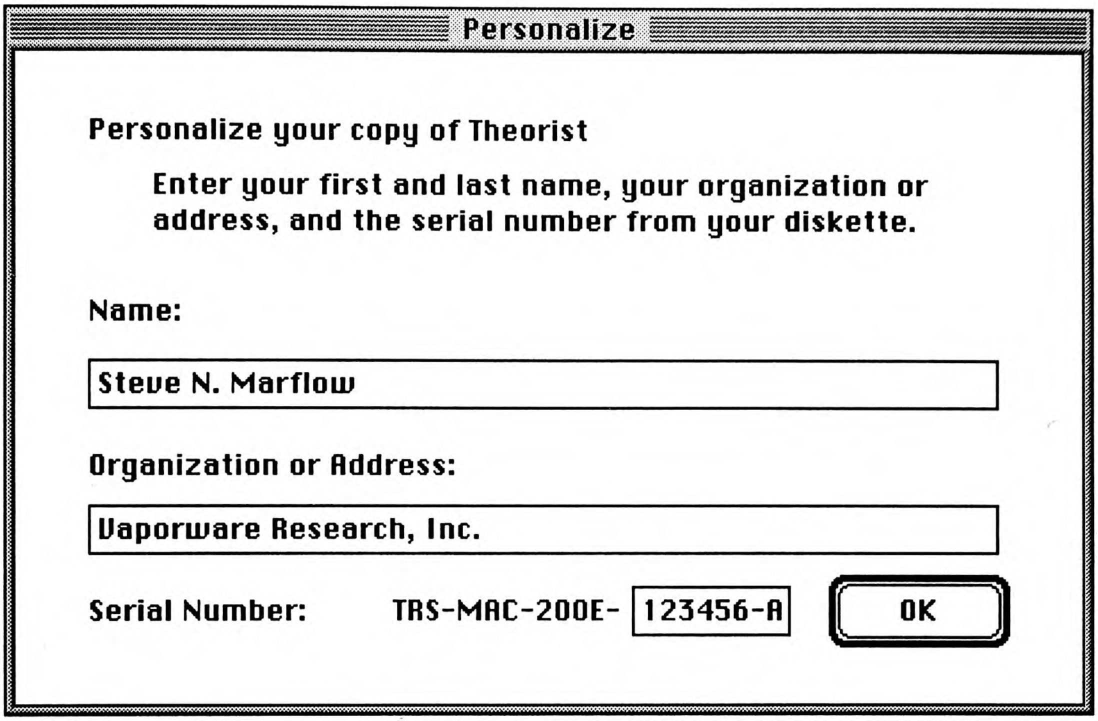
From now on this manual refers to "the Theorist application," which can mean any edition from the three application disks, depending on which one you installed. Ignore minor differences such as the file name (e.g., Theorist®, Theorist®/FPU, or Theorist®/PowerMac).

Personalizing Theorist

Double-click on the Theorist application icon (which is now in your hard disk's Theorist Folder):

The first time you open Theorist, a dialog appears for personalizing your copy of Theorist. Enter your name, organization, and serial number (which is found on the Standard Application Disk) in the places provided. (The first part of the serial number is done for you; you must enter the last eight characters—six digits, a hyphen, and a capital letter—from your serial number label.) When you have completed the information click **OK** or press [return].

You may not leave any of these blank.



After you enter the information and click **OK** another dialog appears and says the personalization was successful and you must click **OK** once more, which causes Theorist to quit. At this point, installation of Theorist is complete and you may begin using it.

If you do not get the "success" dialog, try again, remembering that you must specify the correct serial number, and give your first and last names and an organization name at least six characters long.

Using the Application

To open the Theorist application from the Finder, double-click on the Theorist icon.

Closing the Application

To stop using Theorist, choose **Quit** from the **File** menu. (It may ask you to save open files.)

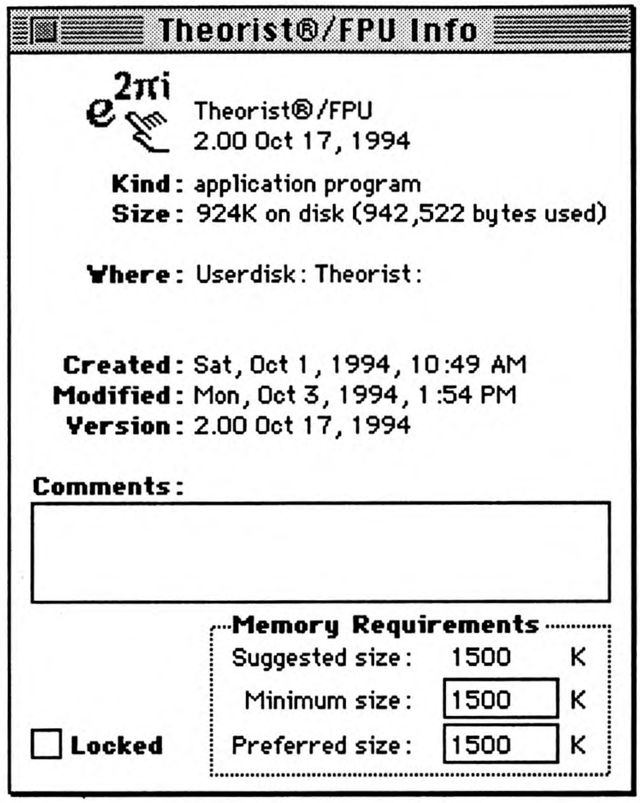
Moving Files and Folders

The New Notebook file and the TheoText.CAT file must be in the same folder as the Theorist application or Theorist will not work properly. You can do whatever you want with the mathematics and graphics folders (e.g., rename, move, delete), but to avoid confusion it is best to leave them as they were installed.

Increasing Application Memory Size

This dialog looks slightly different under different System versions.

To create large notebooks and QuickTime animations, you must increase the Theorist application memory size. To do this, select the application in the Finder and choose **Get Info** from the **File** menu to bring up the Info dialog.



The box in the lower right hand corner specifies the memory to be used by the application. Enter the desired number where it says "Current size" or "Preferred size".

Shown here is 3000K (three thousand kilo-bytes), which is 3Mb (three megabytes).

INSTALLATION PREPARATIONS FOR WINDOWS SYSTEMS

The following preparation instructions apply to computer systems equipped with Microsoft Windows Do not follow these instructions if you are using a Macintosh computer.

Disk Backup

Before you use the Theorist disks, you should make backup copies in case of errors or accidents which destroy the files on the disks. Your Windows and/or DOS manuals explain how to copy a disk. After you have made copies, put the original Theorist diskettes in a safe place and do not use them; use only the backup copies.

Serial Number

Please take note of the serial number on the disk labeled Install 1. This number is needed to install the program and is used when upgrading. Do not give your serial number to others. Write it in your manuals.

System Requirements

Theorist must be used on an IBM PC compatible computer with an 80386 or later CPU running Microsoft Windows version 3.1 or later in enhanced mode. The machine must be equipped with at least four megabytes (4Mb) of memory and a hard disk, which should have at least 2Mb of free space for Theorist and its related files.

Theorist also requires DOS version 3.1 or later. (To check your system version, enter **VER** at the > DOS prompt.)

Best performance is achieved with an 80386 (with math co-processor) or 80486 or better system with four megabytes of memory and a VGA color monitor.

Symbol Font and Greek Letters

In order for you to use Greek letters and other symbols in your equations, you must have the Symbol font installed for use with Windows. The Symbol font is supplied with all Windows versions released by Microsoft.

Follow your Windows documentation to learn how to install it on your system, if it is not already installed.

INSTALLING THEORIST ON A WINDOWS SYSTEM

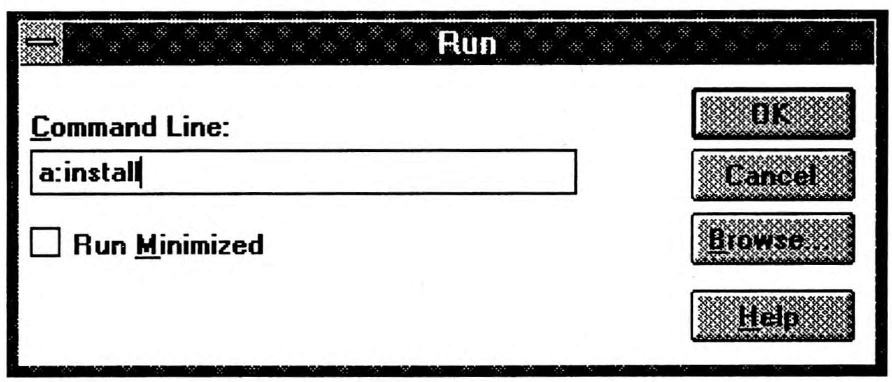
The following installation instructions apply to computer systems equipped with Microsoft Windows Do not follow these instructions if you are using a Macintosh computer.

These instructions assume the hard drive is the C: volume and the 3.5" floppy drive is the A: drive. Substitute other letters as needed.

Insert the disk labeled Install 1 into the 3.5" floppy drive.

Starting from Windows

With your computer currently running Windows, close any open applications. From the Windows Program Manager's **File** menu, choose **Run** to open the Run dialog. In the **Command Line** box type **a:install** or **b:install**, depending on which drive you are using.

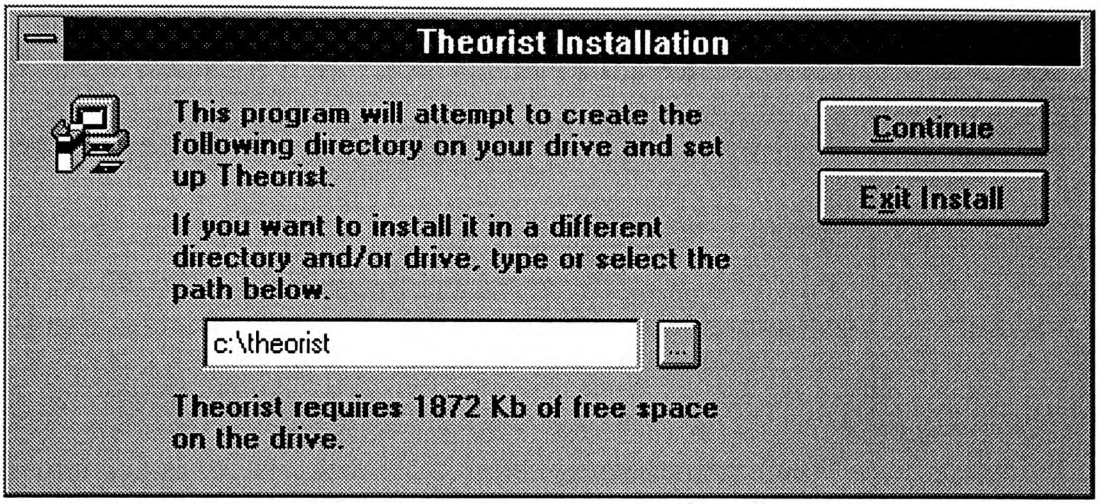


Click the **OK** button to start the Theorist installation program. This takes a few moments.

If you encounter a problem or become lost or confused during this installation, just accept all the default options along the way (e.g., click OK or the highlighted buttons) and everything will turn out fine.

Using the Install Program

The Install program starts up and shows the following dialog.



The installer will create a new directory named Theorist at the path specified in the dialog. (If you want a different name or a different directory, type a new pathname in the box.)

Exit Install

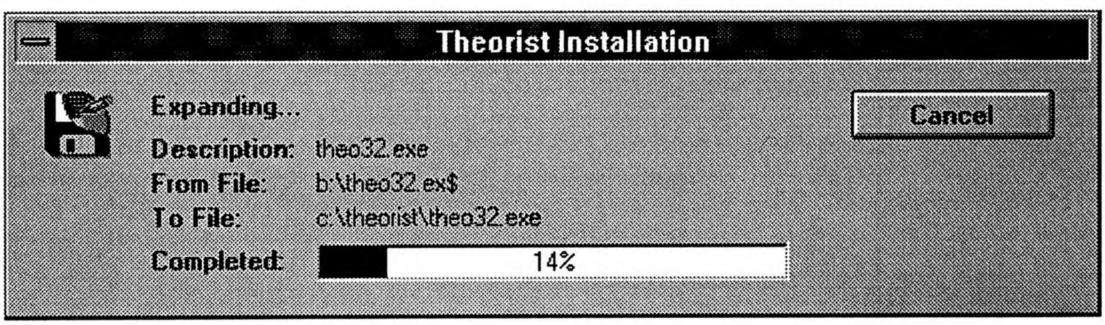
If you do not wish to install Theorist at this time, choose **Exit** to leave the Install program and return to the Windows Program Manager. No files are installed if you do this.

Continue Install

Click **Continue** to install Theorist.

Copying Files

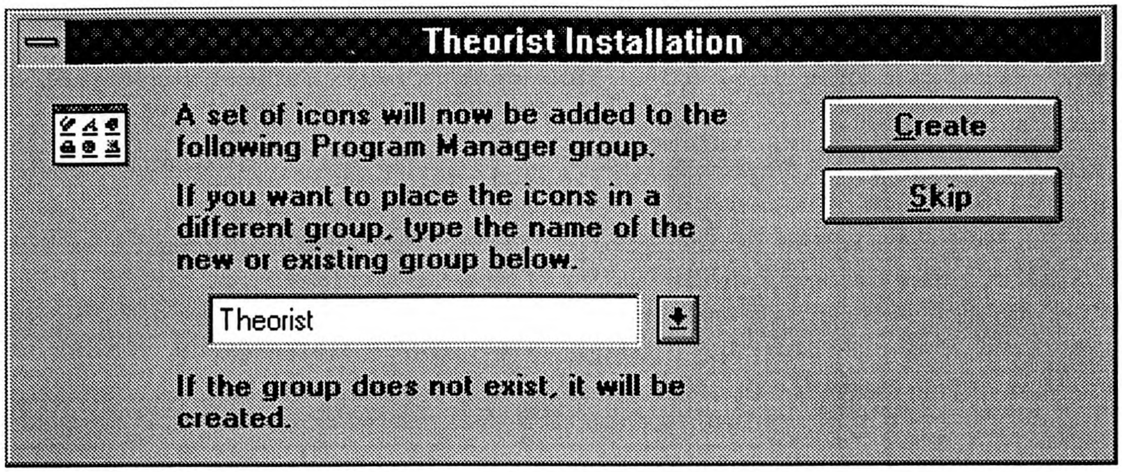
When you continue with the installation, the Install program decompresses and copies various files to the specified directory and informs you of its progress.



Creating a Group

You may type another name for the group.

After the files are copied to your hard disk, the Install program asks whether you want to create a Theorist group or skip it.



Unless you are re-installing Theorist and already have an Theorist group, click **Create**.

Win32s Setup

Next, depending on your computer system, the installer runs the Microsoft Win32s setup program. Follow the instructions.

Exiting the Installer

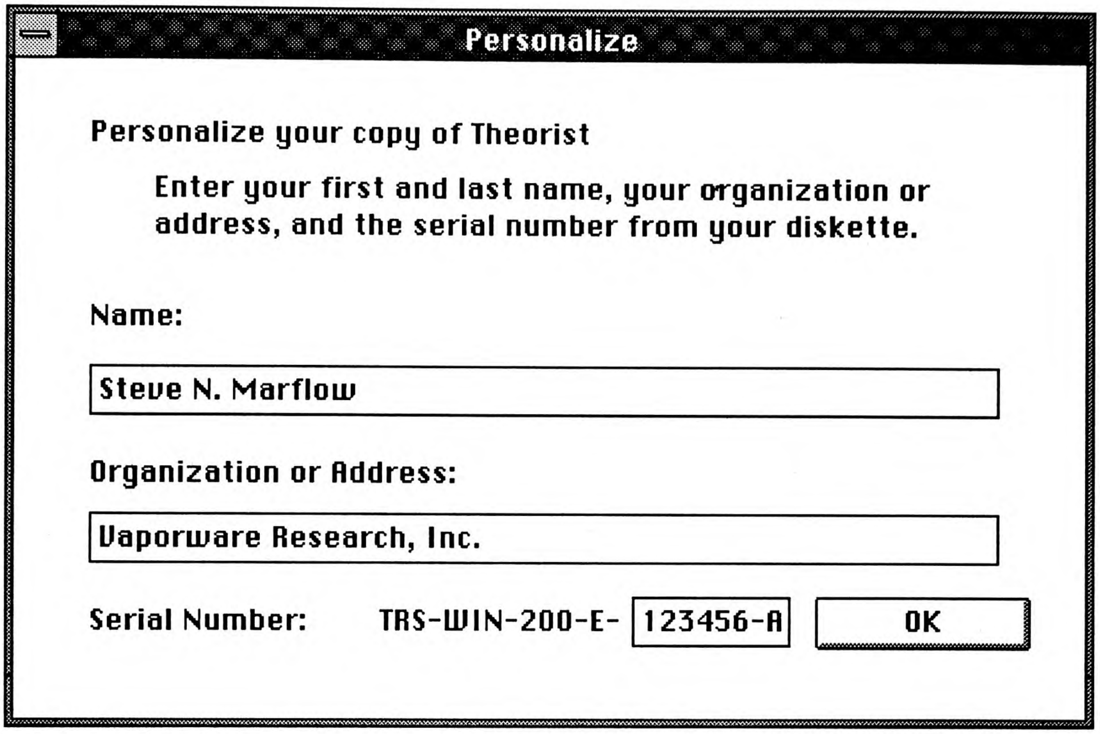
When the installer is finished, it returns to the Windows Program Manager, where the Theorist group is open for you.

Personalizing Theorist

Double-click the Theorist application icon from the Program Manager to run Theorist.

The first time you open Theorist, a dialog appears for personalizing your copy of Theorist. Enter your name, organization, and serial number (which is found on the Install 1 disk) in the places provided. (The first part of the serial number is done for you; you must enter the last eight characters—six digits, a hyphen, and a capital letter—from your serial number label.) When you have completed the information click **OK** or press [return].

You may not leave any of these blank.



After you enter the information and click **OK** another dialog appears and says the personalization was successful and you must click **OK** once more, which causes Theorist to quit. At this point, installation of Theorist is complete and you may begin using it.

Using the Application

To open the Theorist application from the Program Manager, double-click the Theorist icon.

Closing the Application

To stop using Theorist, choose **Exit** from the **File** menu. (It may ask you to save open files.)

Moving Files and Directories

The files NewNote.The and TheoText.CAT must be in the same directory as the Theorist application or Theorist will not work properly. You can do whatever you want with the mathematics and graphics directories (e.g., rename, move, delete), but to avoid confusion it is best to leave them as they were installed.

PRELIMINARIES

The tutorials in this manual follow a progression from simple to complex. In general, each tutorial describes a particular problem then shows you step by step how to solve the problem using Theorist. If you are inclined to a less structured approach, feel free to work with the program independently. If this is your approach, you will find pointers to useful information in the index to this manual and the Reference Manual. If you need help, use Theorist's on-line help system (Balloon Help on the Macintosh). Good luck and happy learning.

Anomalies

This manual uses screen shots of windows and dialogs which may have slightly different borders and controls than windows on your screen. There also may be minor font differences between this manual and your Theorist, including the fonts used in equations. The arrangement of terms in expressions may be different (even though mathematically equivalent). Such differences are of no concern, as they do not affect your use of Theorist.

New Notebooks

Each tutorial assumes you are starting fresh with a new notebook. To create a new Theorist notebook, choose **New Notebook** from the **File** menu. Some tutorials also make use of special rules and declarations in the New Notebook file supplied with Theorist. This file (named NewNote.The for Windows or New Notebook for Macintosh) must be in the same directory as the Theorist application program or Theorist will not have the special rules when it creates a new notebook.

Basics

When you launch Theorist, the program opens a blank, untitled document, a Theorist notebook. Start each tutorial with a new notebook with the default settings. To create a new notebook with the program already running, choose **New Notebook** from Theorist's **File** menu.

Entering Equations

You enter equations in Theorist by typing at the keyboard. (You can also use an on-screen palette to enter some parts of an equation.) If the manual says, "Type: **2*x+y-z**", type in the characters from the keyboard. Characters in this bold typeface, **x*y** for example, are for you to enter.

For example, if the manual states:

Enter: **2*x=y** [return] **(z^3)=2*x**

Type the characters, but press the Return key (Macintosh) or Enter key (Windows) between the two equations.

Special characters are often indicated by the keystrokes used rather than the resulting character. For example, to create a lower case Greek Gamma (γ), you press the ['] key followed by the [G] key. The manual may indicate this with the following:

Enter: **2*'g=y**

Note that **'g** produces a different character than **'G** so be sure to use Shift for upper case when called for.

You are expected to use the Shift key, without instruction, for characters such as *, ^, (, and).

The following table shows how a keyboard formula may appear in the manual, how to type it, and the result it has in Theorist.

Some of these keystrokes are for Macintosh ([option] and [⌘]), while some are for Windows ([alt] and [Ctrl]).

Formula	How to Type It	Result
g	Press the [G] key	*g*
G	Hold down [shift] as you press [G]	*G*
George	Press [shift][G], then [E], [O], [R], [G], and [E]	George
'g	Press ['] then [G] (*no* Shift!)	γ
'G	Press ['] then hold [shift] and press [G]	Γ
[alt][Ctrl][G]	Hold [alt] & [Ctrl] as you press [G] (*no* Shift)	γ
[option][G]	Hold [option] as you press [G] (*no* Shift)	γ
[⌘][G]	Hold [⌘] as you press [G] (*no* Shift)	*Graph*
[Ctrl][G]	Hold [Ctrl] as you press [G] (*no* Shift)	*Graph*
x^y [esc] **z**	Press [X], [shift][6], [Y], [esc], and [Z]	$x^y z$
2 [space] **x**	Press [2], then [space], and then [X]	$2x$
[⌘][/]	Hold [⌘] as you press slash key	*Division*

Where this manual refers to the Return key, it also means the Enter key on IBM PC compatibles.

Do not press Return or Enter unless you are specifically told to do so. If you have used command-driven or line-oriented software programs before and you are accustomed to pressing Return or Enter at the end of an input line in order to execute the command, you must unlearn this habit. Pressing Return in Theorist creates a new expression. Pressing Enter on the keyboard's numeric keypad has another, entirely different effect. Remember: do not press Return or Enter at the end of a line.

The tutorials do not always use the fastest method for entering equations. Instead, the most easily understood (and remembered) techniques are displayed. For shortcuts and command key equivalents, see the Reference Manual.

Theorist uses most of the standard text-editing conventions. However, there are certain differences between Theorist and other software programs. These may be surprising at first, but are designed to improve your efficiency as you work with WYSIWYG equations. For example, the Escape key moves the cursor out of the current selection to the next hierarchical level in the expression. You make selections in the normal fashion by dragging the mouse pointer, but you can also make multiple selections by holding the Shift key and dragging over additional selections. The arrow keys move the cursor and the Tab key also changes the selection, when possible. The Delete (or Backspace) key generally erases the current selection or the character to the left of the cursor. However, Delete also changes the current selection. Be careful typing Delete repeatedly; you may lose large pieces of expressions. (Note: if your keyboard has a ⌦ key it is not the Delete/Backspace key so do not use it as such.) If you make a mistake entering a formula, try the **Undo** command from the **Edit** menu (or click the  button on the palette).

While you are learning Theorist and working through these tutorials, you may want to delete an entire incorrect equation (rather than trying to edit it) and re-type the whole equation.

QUICK SOLVE AND GRAPH

This tutorial provides a very quick introduction to entering an equation, manipulating it, and creating a three-dimensional graphic representation of the original equation.

Launch Theorist by double-clicking on its application icon in the Finder (Macintosh) or Program Manager (Windows).

Type **z=1.5^x** [esc] ***sin(y)**, which gives you:

$$z = 1.5^{x}\sin(y)$$

Select the *x* by dragging over it. Choose **Isolate** from the **Manipulate** menu to solve the equation for *x*.

$$x = \log_{1.5}\left(\frac{z}{\sin[y]}\right) \quad \textit{Isolate}$$

Notice Theorist shows the manipulation step name to the right of the resulting equation.

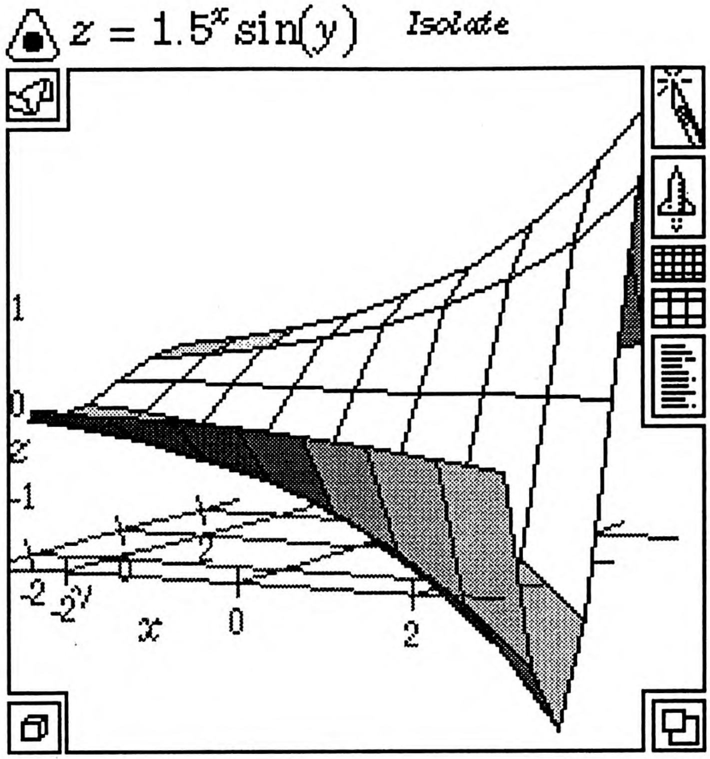
Select the *y* in the result. Choose **Isolate** from the **Manipulate** menu to solve the equation for *y*.

$$y = \arcsin\left(\frac{z}{1.5^{x}}\right) \quad \textit{Isolate}$$

Select the *z* in the result and, once again, choose **Isolate** from the **Manipulate** menu to return the equation to its original form.

$$z = 1.5^{x}\sin(y) \quad \textit{Isolate}$$

With the cursor (or any selection) anywhere in the latest equation, pull down the **Graph** menu and move the pointer to the **z = f(x,y)** item, which drops down a submenu. From this submenu, choose **Illum. 3D**. A dialog box appears to verify the axis variables; click **OK** to accept the default assignments. Theorist creates an Illuminated three-dimensional graph of the equation.



To rotate the display, point to the graph surface with the mouse (the pointer becomes a hand). Click and hold down the mouse button and drag the graph surface to the left or right. Notice how the graph moves with the hand. Try dragging the image in a complete circle. Or drag the graph so you see it from the top.

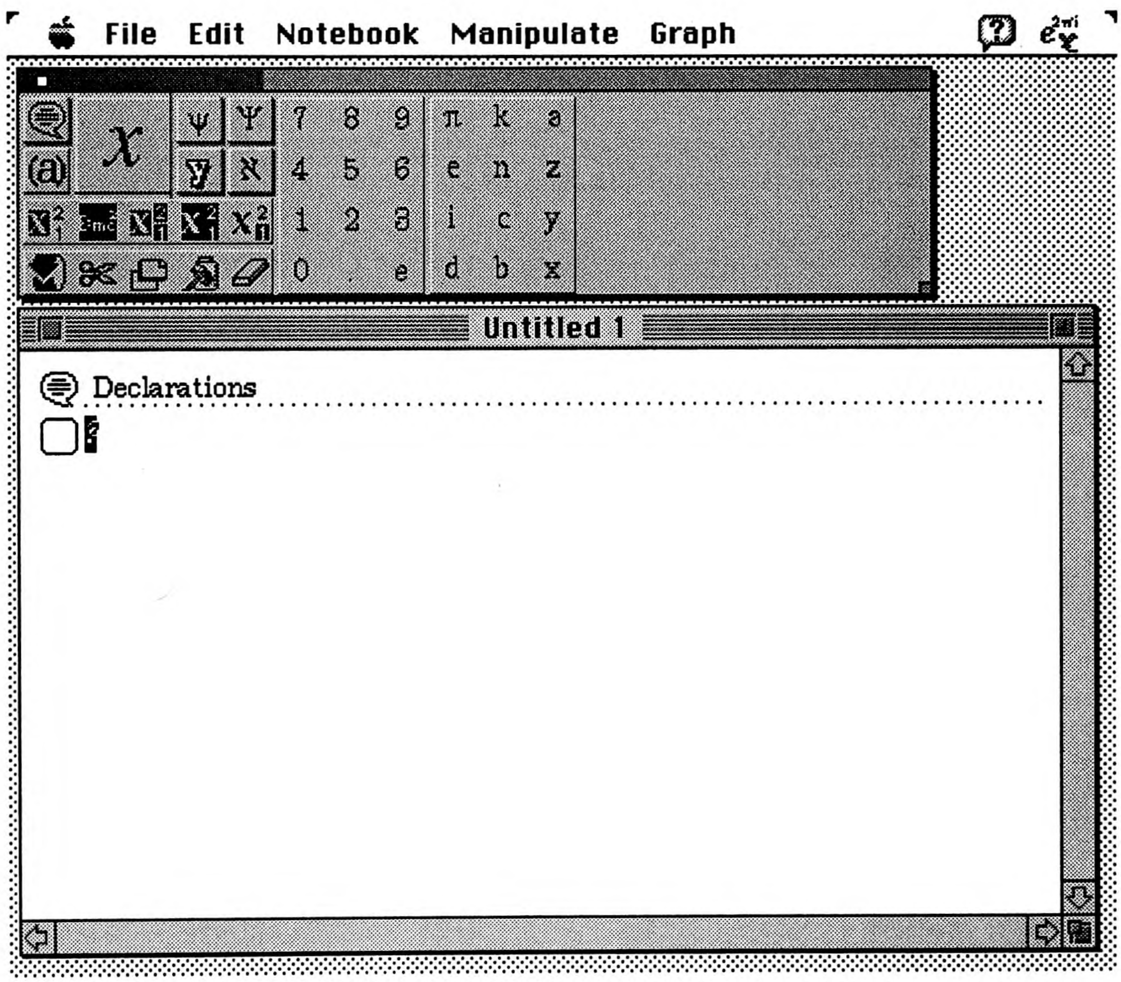
Notebooks

Basic Concepts

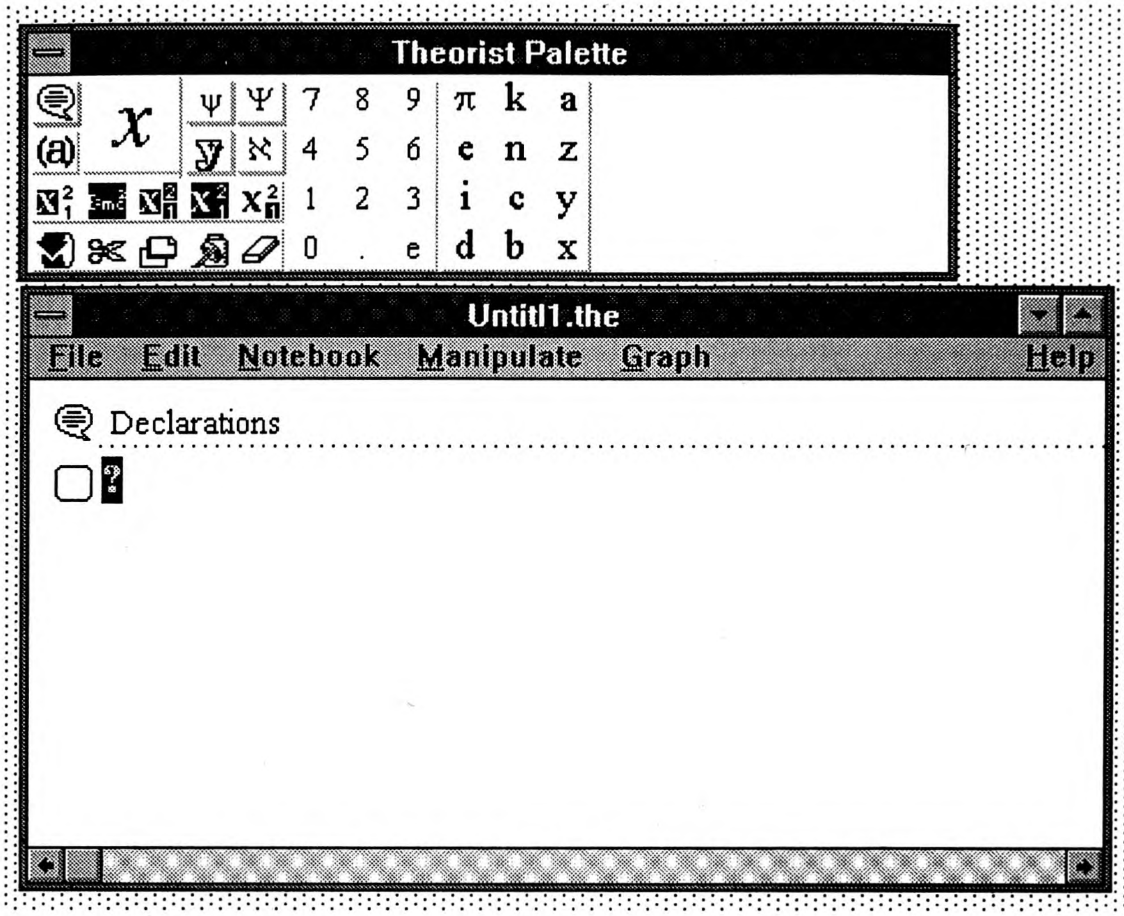
This chapter describes some basic concepts in Theorist including:

- The components of a Notebook, Theorist's file format
- How to enter equations from the palette and from the keyboard
- How to select parts of an expression
- How to edit expressions

When you start up the Theorist application, the screen looks like the following on a Macintosh.



Under Microsoft Windows, when you start up the Theorist application, the screen appears like the following.



The large window—named "Untitled 1" and "Untitl1.the" in these examples—is your notebook, or work area. It is analogous to a word processor document or a spreadsheet program file but is perhaps most like an outliner document.

On the above screen, this window shows two Theorist "propositions." The first one, "Declarations," is a comment, a heading for the notebook. You can type anything into it and it is ignored by Theorist in calculations. Go ahead and do this now: click the pointer anywhere in the word "Declarations" and when the blinking cursor appears, start typing. (If you are wondering why it is named Declarations and what the dots below it are for, this particular comment has some special declarations hidden inside it, but we will discuss this later in the "Name Declarations" tutorial.)

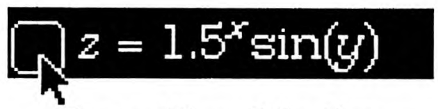
The second proposition is an assumption—an equation that you type to tell Theorist something. Notice that the comment has a quotation bubble icon to its left, whereas the assumption has a rounded-corner square icon. These proposition icons represent two types of propositions. There are many other types, discussed later in this manual.

The Return key on most IBM PC type keyboards is labeled Enter instead of Return.

You can make more assumptions by pressing Return and you can make more comments by pressing Enter on the numeric keypad. Press [enter] now and a bubble icon appears with a blinking cursor. Type "hello". Press [shift][return], which adds a line to the current comment. Type "world".

Use the mouse pointer to select the word "world." You can either drag over it, double-click it, or Shift-click it to make the selection. Press [delete] (or Backspace) and notice the selection disappear.

You can select propositions by clicking on their icons. A proposition selection is different from an equation selection and manipulations have different effects on these different selections. Press [return] and type **z=1.5^x** [esc] ***sin(y)**. Click on the equals sign to make an equation selection. Click on the square assumption icon to make a proposition selection. Notice the difference in the way the two selections are highlighted.



Proposition Selection

Equation Selection

You can delete a proposition by selecting it and pressing Delete on the keyboard.

Entering Equations

Theorist gives you two methods of entering expressions and equations: keyboard entry and palette entry. Keyboard entry consists of typing formulas as in a programming language or spreadsheet; it may be hard to learn for some, but it is fast. Palette entry lets you use the mouse to click on palette buttons on the screen that represent various symbols and operators. Both methods are reasonably fast, but the keyboard method is faster for experienced users.

Palette Entry

Theorist has a rectangular palette at the top of the screen for entering equations. Like a window, you can drag it to other locations on the screen and you can have it showing or hidden. Actually, it is two palettes in one. The variables palette, labeled with the big x, is for entering names for variables and constants. The functions palette, labeled with the big $f(x)$, is for entering operators and functions. Clicking on the big x or $f(x)$ buttons toggles between the variables and functions palettes. Do this now and notice the palette switching between the two sides.

There is a preference option to prevent the palette from switching automatically.

When you click on a palette button, the desired symbol, function, or operator appears in your notebook and the palette switches sides, anticipating your next entry. For example, click on the a button from the variables palette; it switches over to the functions palette. Click the on the exponent button a^c and the palette toggles back to the variables. Click the y button. The process continues. What you just clicked is the palette equivalent of the keyboard formula **a^y**.

The palette method of entering expressions is affected by a special option, described below.

Keyboard Entry

Fortranish is similar to other programming languages and spreadsheets, too. The significant difference between Fortranish and non-Fortranish is the escape level behavior.

Theorist has two styles of entering expressions from the keyboard. The default style works in much the same way expressions are entered into Expressionist, if you are familiar with that program. The other style, 'Fortranish,' works like—you guessed it—the FORTRAN programming language.

The two styles are very similar. You can type in expressions in a manner similar to the way you would type them into a computer spreadsheet or into a Pascal program. Addition and subtraction are represented by the familiar "+" and "–" characters; multiplication is "*" and division is "/". For a complete listing, see the Theorist Reference Manual.

For example, try typing **(a^2)=(x^2)+2*x*y+y^2**, which appears in your notebook as:

$$\square \left(a^2\right) = \left(x^2\right) + 2xy + y^2$$

Choose **Clarify** from the **Notebook** menu to eliminate the unnecessary parentheses, which gives you:

$$\square\ a^2 = x^2 + 2xy + y^2$$

Notice that Theorist figures out what to do as you type. For instance, when you type **(x^2)+2** it gives you $x^2 + 2$. To change this grouping, type as you would in a spreadsheet, without parentheses. If you want x^{2+2}, type in **x^2+2**. Try this by pressing [return] and typing **y=x-1/x+1**. It looks like:

$$\square\ y = x - \frac{1}{x+1}$$

But if you press [return] and type **y=(x-1)/x+1**, you get:

$$\square\ y = \frac{(x-1)}{x+1}$$

Notice that as you type the expression, the "(" makes both the opening and closing parentheses ( ). When you type the ")" closing parenthesis, you are telling Theorist you are done typing the contents of the parentheses and it places the blinking cursor to the outside of the parentheses. With this scheme, you never have mismatched parentheses.

Sometimes, Theorist changes parentheses to brackets and braces, and vice versa. Do not be alarmed; this has no effect on how the equation is manipulated; it only affects the screen display. You may even type []'s, {}'s, and ()'s at will, as they are interchangeable. Internally, Theorist knows what is enclosed in what and never needs to see the parentheses to understand it. They are there for your benefit in your efforts to enter expressions.

Theorist removes parentheses that are unneeded, especially in exponents, subscripts, and the tops and bottoms of fractions. Even though Theorist does not always show parentheses in these, you generally have to use parentheses while typing them in or Theorist misinterprets your meaning. When you need to enclose something in parentheses after it has been typed in, select it whole and type an open parenthesis: "(".

In many cases, you can enter expressions without the unneeded parentheses. The Escape key moves the selection (the blinking cursor) so parentheses are not necessary. For example, if you wanted to create the expression $x^2 + 2$ with the Fortranish option off, you could type **(x^2)+2** or **x^2** [esc] **+2**. Notice how as you type the Escape key, the selection moves from the superscript area back down level with the x.

Fortranish

You have seen Theorist accepts keyboard formulas with the Fortranish option off, which may be the preferable choice for new Theorist users and experienced Expressionist users. However, people accustomed to entering formulas in spreadsheets and programming languages may prefer Fortranish keyboard entry, which we will explore now to show its differences.

Press [return] and type **x^2+2**. Notice how the + 2 stays up at the exponent level.

$$\square \; x^{2+2}$$

Choose the **Fortranish** item from the Notebook menu's **Preferences** submenu. Again, press [return] and type **x^2+2**. Notice how the cursor automatically drops to the x level for the + 2.

$$\square \; x^{2}+2$$

With the Fortranish option on, re-enter the other equations and notice the differences.

Hiding the Palette

When you become proficient at keyboard entry, you may want to remove the palette at the top of the screen. Choose **Palette** from the **Notebook** menu's **Windows** submenu to un-check the menu item and hide the palette. (This also gives you more room for the notebook window.)

There are more details to entering complicated equations; examples in the following tutorials show you how. The Theorist Reference Manual has everything you need to know about entering equations into Theorist in its Expressions and Editing sections.

Exercise

Try to create the following equation by first using the palette, then using non-Fortranish keyboard entry, then a Fortranish formula.

$$a = \frac{1}{\frac{1}{x} + \frac{1}{y}}$$

Hint: It does not need to look exactly the same, just as long as it has the same meaning. Effective use of parentheses is the key to success.

If you find you can construct it without difficulty using a particular method, consider using that equation entry method from now on.

Selecting

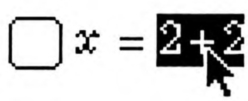
You make a selection so that you can then instruct Theorist to change, delete, or otherwise modify the selection by some action. Most manipulations and commands need a selection in order to work.

As you may have realized by now, you can select any part of any equation or expression simply by clicking and dragging with the mouse. However, you can not select an op, such as = or +, by itself; you must include the terms it operates on, such as $x = y$ or $a + b$.

Selection Shortcuts

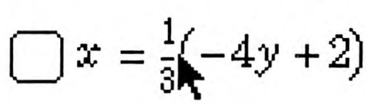
There are shortcuts for selecting with the mouse pointer. For instance, you can select an entire equation just by clicking on its equal sign:

Another shortcut is to select a sum of terms by double-clicking on an addition sign in the expression:

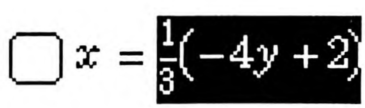


You may select integrals by clicking on an integral sign:

Double click here and drag to the right just a bit:



...to select this term:



Select cross products by clicking on the cross product symbol:

$$\square \; x \times y = z$$

By now you should have an idea of how selection shortcuts work. The general rule of thumb is to click or double click on the symbol or operator. Try experimenting with selection shortcuts by creating large equations and clicking various locations within the equation.

Multiple Selections

You can make multiple selections by Shift-clicking, which involves holding down the Shift key as you click down. Try making two or three selections in different equations in your window. Then, try typing something in. Whatever command you choose or whatever you type affects all the individual selections simultaneously.

Note: You can't make multiple selections in comments.

Every time you click without Shift, you start another selection, deselecting your previous selection. After the first, every Shift-click adds to the set of selections. If you try to add another selection that is inside (or another part of) an existing selection, you de-select both.

Selection Tools

The palette has five buttons especially for changing the selection. Let's enter an expression to explore these selection tools. Make a new assumption (press return) and click the $\sum_{a=b}^{c}$ palette button, then fill in the question marks to make the following equation. (After entering the summation, you can type **k** tab **1** tab **4** esc **x^k** esc **=y**.)

$$\square \; \sum_{k=1}^{4} x^{k} = y$$

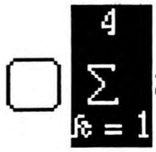
Click the equation's equal sign to select the whole equation. Click the Select In button to make the following multiple selection.

$$\square \; \sum_{k=1}^{4} x^{k} = y$$

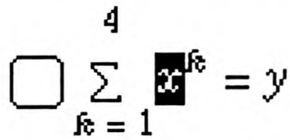
Select the *k* in the summation index. Click the Select Next button repeatedly and notice the selection change. Press Tab or type a colon to do the same thing.

$$\square \; \sum_{k=1}^{4} x^{k} = y \qquad \square \; \sum_{k=1}^{4} x^{k} = y \qquad \square \; \sum_{k=1}^{4} x^{k} = y$$

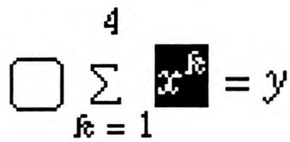
(Leave the 4 selected.) Press esc and click Select Next repeatedly and notice the changing selection.



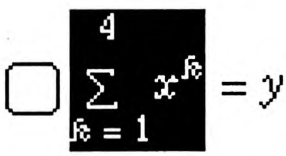
(Leave the x^k selected.) Click the Select First button to select the first element in the current selection, which happens to be x.

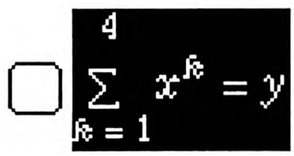


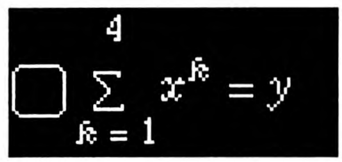
Click the Select Out button.



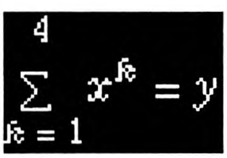
Click Select Out twice again and notice the changing selection.

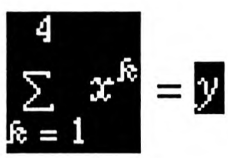


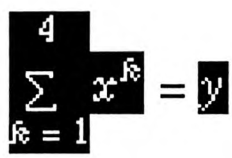


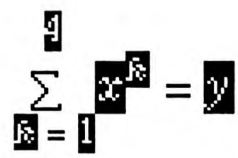


Click the Select In button repeatedly and notice the selection change each time.

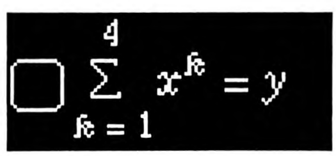








Click the Select Proposition button and the entire proposition (including its icon) is selected.



See Appendix A for a list of keystroke equivalents for these selection tools.

You should be able to select just about any individual character or expression with these selection tools, but there is always the standard click and drag method for when they can not help you.

Modifying Equations

Modifying equations can seem to be somewhat counter-intuitive.

If you are trying to delete something, it is best to select it, and press Delete one or more times to get rid of it. Sometimes it takes a few deletions; frequently Delete leaves a question mark as a place-holder, which gets deleted with the next keystroke. As you continue to type Delete, it gradually unravels the whole equation or expression.

Create a new notebook (with Fortranish off). Type **y+z+3x^2**. Select the 2. Press [delete]. Keep on pressing [delete] until you have nothing left. Repeat as often as necessary in order to understand it.

If you are trying to replace something, it is best to select what will be replaced, and then type in the new expression.

Create a new assumption (press [return]) and enter the same expression again. Select the 2 and type **5+n**. Select the *x* and type **sqrt(2)**. Select the $y + z$ and type **sin(x)**.

$$\square\ \sin(x)+3\sqrt{2}^{5+n}$$

When augmenting an expression with new parts, the new parts almost always start with an operator. In this case, select what you want to enclose in the op and then type the operator or click it from the palette.

Create a new assumption and type **3x^2**. Select the 3. Type **+a**. Then, press [delete] twice to get rid of it.

Select *x*. Type **+a**, then press [delete] twice to get rid of it.

Select the 2. Type **+a**, then press [delete] twice to get rid of it.

When you replace a negated expression, you must press Delete before typing the new expression. For example, select the 2 and type **-a**. press [delete] *three* times to get rid of it. Two Deletes are not enough to get rid of the negation.

NOTEBOOK STRUCTURE

This chapter describes:

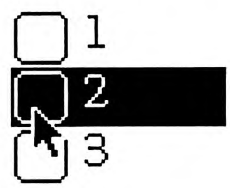
- Using Theorist's outlining tools
- Creating and editing Comment propositions
- The Notebook Font

Outlining

Theorist allows you some degree of flexibility in arranging the ingredients of notebooks.

Create a new notebook and type **1** into the empty assumption. Press [return] and type **2**. Repeat this process through ten.

Click on the second proposition icon so it is selected.

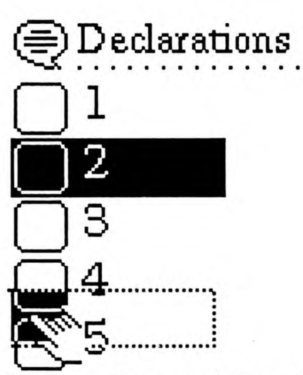


Dragging a Proposition Outline with the Hand Cursor

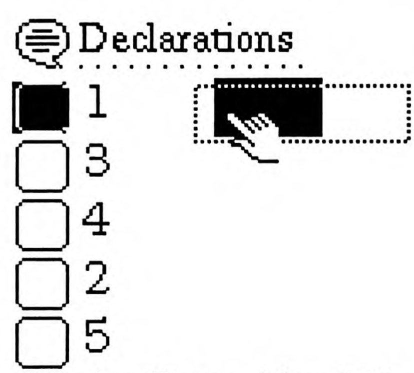
With the pointer over the icon, hold down the [⌘] key (Macintosh) or [Ctrl] key (Windows). Notice how the pointer turns into a pointing hand. With [⌘] or [Ctrl] held, click and drag the second icon about the screen. (Once you start dragging, you may stop holding the key.) Note that an outline of the proposition is dragged around.

Also notice that at certain locations on the screen, highlights appear under the hand. These are locations where you can drop whatever the hand is holding. Most places between propositions can accept a new proposition. If a location does not highlight, you may not place anything there.

Drop 2 between 4 and 5. Then take 10 and move it to the right of 1.



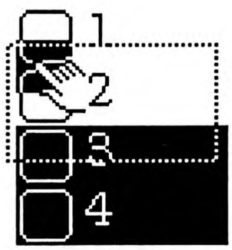
Moving Proposition #2



Moving Proposition #10

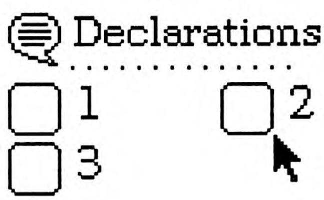
Now put them all back in order again.

Now that everything is back in order, we will move multiple propositions at a time. Select 3 and 4. To do this, click 3's icon to select it, then hold the Shift key as you click 4's icon to select it. Hold down the ⌘ or Ctrl key and drag 3 and 4 (as a group) to a new location between 1 and 2.



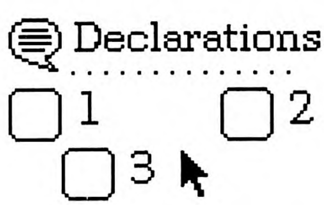
Once again, rearrange all of the propositions so they are back in order.

Take the second proposition and move it to the right of the first.

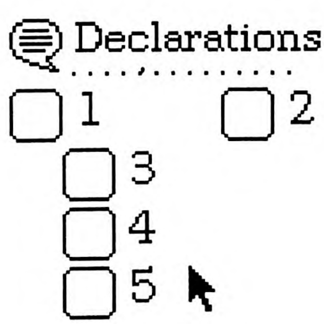


Every Theorist proposition can have another to its right, called an Inline proposition. (More properly, 2 is an 'inline' of 1.) This is useful, for instance, for putting several short equations all on one line, or for putting two graphs side by side.

Now grab proposition 3 and move it up slightly and over to the right so it is indented below 1.

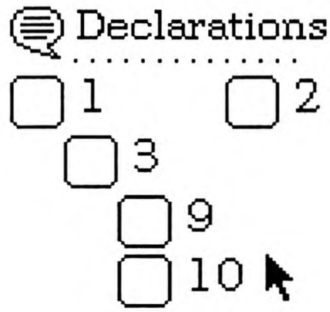


In this configuration, 3 is a 'daughter' of 1. Now, move 4 directly under 3, and 5 directly under 4.



Configured like this, 3, 4, and 5 are all daughters of 1. They are also referred to as 'sisters' of one another. In fact, 1 and 6 through 10 are all considered sisters of each other.

Next, make 9 a daughter of 3 and move 10 immediately below it:



Daughters can go on for as many levels as you can fit on your screen, and beyond, with the aid of the scroll bars.

Now click on the first proposition icon and its daughters are also selected. Choose **Collapse** from the **Notebook** menu (or double-click the selected icon). All of 1's daughters collapse under 1. (Note that 2, a sister of 1, is still visible.)

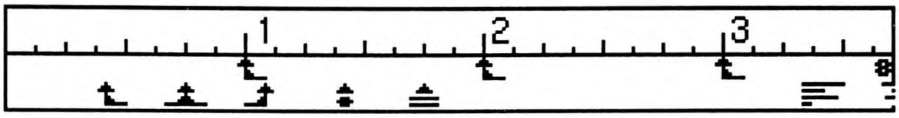
With the first proposition selected, un-collapse it by choosing **Expose** from the **Notebook** menu. (Or, you can double-click it.)

Theorist's ability to collapse certain information is a good way to hide information. For instance, expose the first proposition in the notebook, "Declarations." The name declarations for different variables appear. (Don't worry about them now. You will explore them at length in future tutorials.) Collapse the declarations.

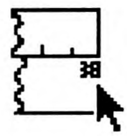
Comments

Press [enter] on the numeric keypad, which makes a special proposition called a comment. You can type normal text into a comment and Theorist does not try to interpret it as an equation. (You can also enter expressions in comments, but they can not be manipulated.) Like a word processor, you can click, drag, delete, et cetera. Type "The quick brown fox jumped over the lazy dogs." Select the word "quick". Pop-up the palette button and choose the long button with the font name, which is analogous to **Comment Font...** on the **Edit** menu. In the resulting dialog choose a different font family and click **OK**. In the sentence, select the word "brown". Choose **Comment Font...** again and enter a larger absolute point size, such as 24, and click **OK**. Select the word "fox" and with the **Comment Style** submenu, make it boldface. Finally, if you are using a color monitor, select the word "jumped" and change its color with the **Comment Color** submenu.

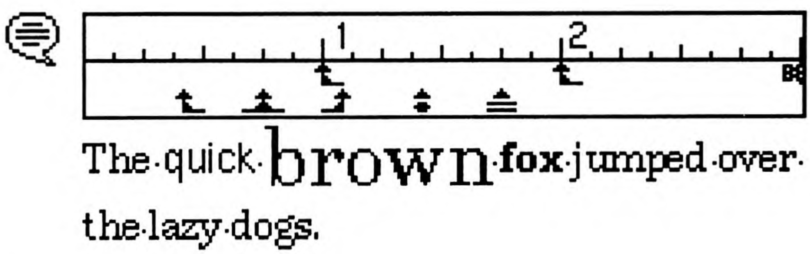
The comment text is formatted according to a ruler, usually invisible. To show all comment rulers in a notebook, choose the icon from the pop-up palette.



Like the ruler in a typical word processor, a comment ruler has different types of tab stops and justification options. However, only the right margin is adjustable. Grab the margin handle on the right of the ruler for our quick brown fox comment,



...and drag it left a few inches so the ruler is approximately half its original size. Notice how the comment text is broken up over multiple lines to compensate for the smaller margin.

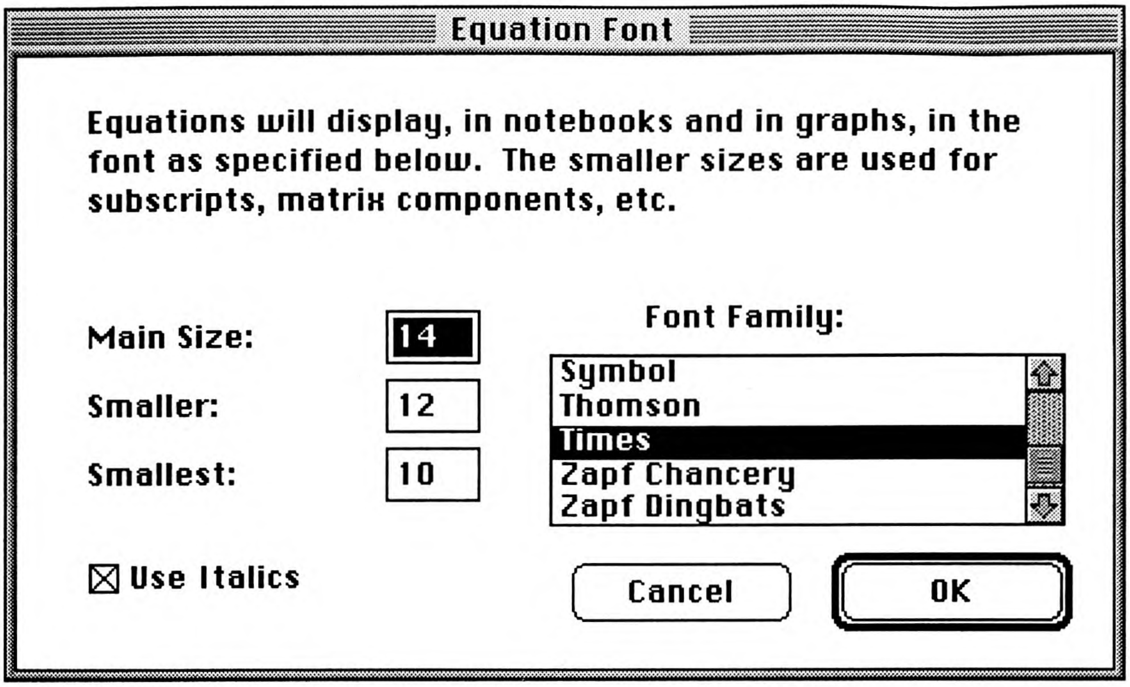


When comment rulers are showing, dots appear where spaces are in the comment text. These dots are a visual aid and do not appear when the notebook is printed.

To hide all comment rulers in a notebook, again choose the icon from the pop-up palette. Ruler settings remain in effect whether the rulers are shown or hidden.

Notebook Font

From the **Notebook** menu, choose the **Notebook Font ...** item. A dialog appears, in which you may use set the default font, size, and italic style.



These font specifications are considered the "default font" for the whole notebook. The Notebook Font affects not only comments, but also expressions in assumptions, conclusions, graph propositions, et cetera. Change the Notebook Font family and sizes to something of your own choice and click **OK**. All propositions change to the new specifications.

Notice the word "quick" remains in the font family you specified earlier, and the bold style is still applied to the word "brown."

When a new Notebook Font is specified, it changes the main comment font but leaves untouched those parts which have had explicit font/size/style instructions.

A Brief Tour of Theorist

This chapter describes

- How to use the Calculate and Simplify manipulations
- How Theorist ReManipulates
- The Theorist Names Variable and Function
- Theorist Propositions
- How to manipulate matrices

Simple Calculations

Here we learn how to do simple calculations with Theorist. We explore the Calculate and Simplify manipulations, as well as some of the central ideas behind Theorist derivations.

Theorist does mathematics mainly in three ways:

❶ Numerically (calculations)
❷ Symbolically (equations)
❸ Graphically (graphs)

Theorist has in it all the functions of a pocket calculator. Therefore it can take any equation you give it and crunch all of the numbers together to get an answer that is a number. The problem with this method is that sometimes you want a more abstract answer, an equation.

Equations can be manipulated symbolically, doing algebra, calculus, et cetera as you would on paper. Theorist can be much more powerful doing things symbolically, but the power comes at the price of more computer overhead (memory and processing speed). If your task is too big or too complicated, you may end up having to do it numerically. Some problems can not be solved symbolically.

Theorist can draw graphs of mathematical relationships. It does this with high-intensity calculations. More on this subject later.

Calculate and Simplify

Type **3+4**.

□ 3+4

Notice that Theorist does not instantly calculate the answer. Theorist is not a pocket calculator; it has a bit more finesse. Theorist is waiting for you to tell it what to do. (Maybe you want to type more into the equation.)

Now, drag to select what you just typed and click (or choose **Calculate** from the **Manipulate** menu). Theorist now calculates the answer like a pocket calculator.

$$\triangle\ 3+4=7 \qquad \textit{Calculate}$$

Select 3+4 again, but this time click (or choose **Simplify** from the **Manipulate** menu).

$$\triangle\ 3+4=7 \qquad \textit{Simplify}$$

Theorist simplifies the 3 + 4 and turns it into 7. You might be thinking that Simplify and Calculate do the same thing. They don't.

Let's calculate something different. Select the 3 in the original expression, and click the $\sqrt{a}$ icon on the palette so the expression looks like this:

$$\square\ \sqrt{3}+4$$

This happens so quickly you may not see the crossing out of the remanipulated expressions.

As soon as you do this, notice how Theorist re-manipulates your answers. First, it crosses out the equations. Immediately after, it changes both of them to look like this:

$$\triangle\ \sqrt{3}+4=5.7321 \qquad \textit{Calculate}$$
$$\triangle\ \sqrt{3}+4=\sqrt{3}+4 \qquad \textit{Simplify}$$

Theorist re-does the manipulations when you change the inputs, like a spreadsheet with automatic re-calculation.

Select the 4, and click the $\sqrt{a}$ button again.

$$\square\ \sqrt{3}+\sqrt{4}$$

Notice the automatically remanipulated Calculate answer.

$$\triangle\ \sqrt{3}+\sqrt{4}=3.7321 \qquad \textit{Calculate}$$

Theorist calculates answers to fifteen digits of precision, but only five digits are shown in this case because our precision display option is set to five digits. If you pull down the **Notebook** menu's **Display Precision** submenu, you can choose the precision you need.

Notice the automatically remanipulated Simplify answer.

$$\triangle \sqrt{3}+\sqrt{4}=\sqrt{3}+2 \quad \textit{Simplify}$$

Simplify simplifies it as best it can. That means simplifying the square root of 4 to a 2, but that also means leaving the square root of 3 alone. Because the square root of 3 is unchanged, Theorist can not add the two numbers together, and leaves the equation unchanged.

Calculate and Simplify are different. Calculate gives numeric answers and Simplify gives answers in symbolic form.

Symbolic answers and numeric answers are useful at different times. For instance, the symbolic answer tells you that if you subtract 2 and square the number, you get exactly 3, to an infinite amount of precision. If you need to cut a piece of wood that length, though, you probably need to use the numeric form.

Variables

Theorist allows variables to represent unknown or unspecified values.

Press [return] to create a new expression and type **x+3*8**.

$$\square\ x+3\cdot 8$$

Select it and try to Calculate it.

$$\triangle\ x+3\cdot 8=x+24 \quad \textit{Calculate}$$

Theorist does not know what value to use for x so it leaves it unspecified. Notice that no truisms have been violated.

You can have definitions of the variables used in calculations. Press [return] to create a new expression and type **x=5*7**.

$$\square\ x=5\cdot 7$$

Select the equation and choose **Make Working Stmt** from the **Notebook** menu. This tells Theorist to look here when it needs a value for x in a calculation, in the event you have more than one equation that could specify the value of x. Once you do this, a dot in the square reminds you that this is a working statement.

$$\boxdot\ x=5\cdot 7$$

Now, if you go back and Calculate $x + 3\cdot 8$, it gives an answer.

$$\triangle\ x+3\cdot 8=59 \quad \textit{Calculate}$$

However, if you use Simplify, nothing happens. Working statements only work for Calculate.

Numeric and symbolic variables are used extensively in later tutorials.

Functions

Variables are only one class of name. Another class is Functions. For instance, the name log is a function that takes the logarithm (base 10) of a number.

Press [return] to create a new expression and type **log(1000)**.

$$\square \; \log(1000)$$

When you Calculate it, you get:

$$\triangle \; \log(1000) = 3 \qquad \textit{Calculate}$$

Theorist has a wide variety of built-in functions, including natural log, trigonometric, and hyperbolic functions, which are explored further in later tutorials. For a complete list of built-in functions, look up "Predefined Names" in the Theorist Reference Manual.

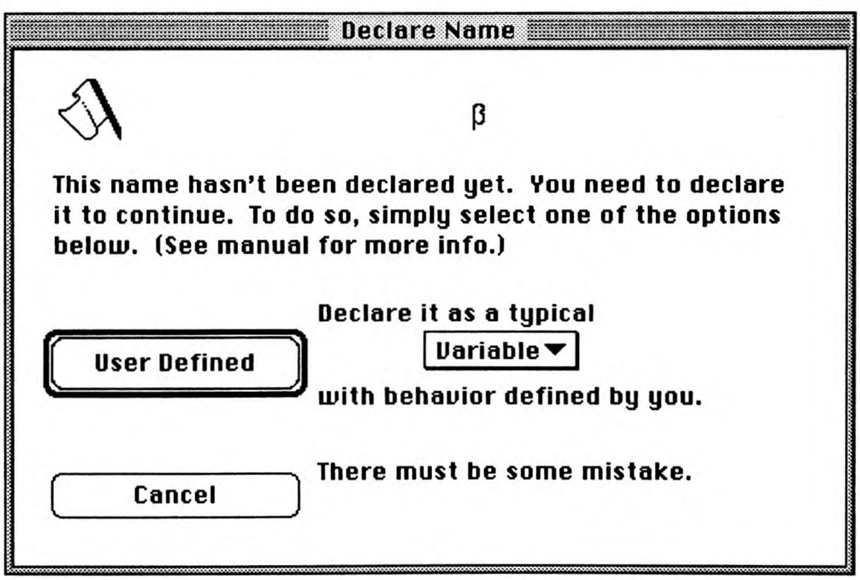
To supplement Theorist's predefined functions, the Special Functions directory (in the Mathematics directory) that comes with Theorist contains more advanced functions.

Undefined Names

The ' b keystroke gives you the Greek letter beta.

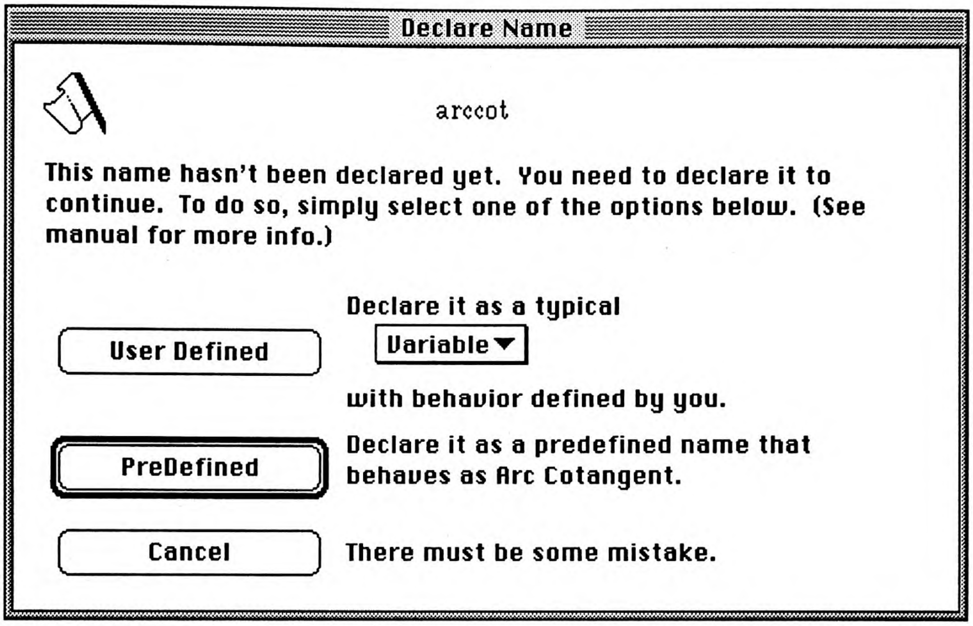
If you use a name not recognized by Theorist, it eventually notices and checks with you to make sure it is not some other name mistyped. If it is a new name, confirm it at this time.

For example, type [return] **'b**. Choose **Clarify** from the **Notebook** menu. Theorist shows you the following dialog.



Theorist asks you to define β You can use the pop-up menu (**Variable** above) to set its class; this is explored in another tutorial.

Press [return] and type the name "arccot" and Clarify again. Since arccot is a name that Theorist recognizes, you get a different dialog.



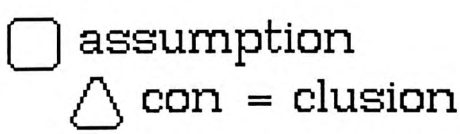
Theorist recognizes the name arccot because it is in Theorist's internal library, and assumes you want it predefined to behave as Arc Cotangent. However, it asks you to either verify this or define it as something else.

If you use a name that you do not intend to have a special, predefined behavior, click on (User Defined). For example, type **a='p*r^2** then Clarify. Theorist asks you to define *r*, but not *a* and π, which are already defined in the notebook's default declarations.

The Name Declarations tutorial explores this area further.

Propositions

You have probably noticed expressions have a square icon to their left, whereas the answers have triangular icons. Square icon expressions are called "Assumptions"; they are expressions or equations that you type in. Triangular icon equations are called "Conclusions". They are always equations; they are always the result of Theorist doing some sort of calculation or manipulation.



Theorist always generates an answer in the form of an equation, even if you start with just an expression. This initial expression is so that everyone remembers the question. This is important. Otherwise, at the end of a session, you would have little fragments of expressions all over the screen and it would be easy to forget what goes with what.

Theorist works with "propositions", which are statements that are assumed to be true. When you calculate an expression 3 + 4, Theorist makes an equation 3 + 4 = 7. When you give Theorist an expression, it converts it into an equation.

In some computer systems, the equals sign defines a name, moves a number to a storage place, or does something similar. For instance, you might have a statement $n = n + 1$. When this statement is executed, it takes the number in storage place n, adds one to it, and restores it in storage place n. In one of these systems, the left side of the equation must always be a variable; $n + 1 = n$ would be an error.

In Theorist, an equal sign means the same thing that it does in normal mathematics: it is a statement that two expressions on either side of the equal sign are the same, as in the statement 3 + 4 = 7. If you turn it around backwards, it means the same thing: 7 = 3 + 4. In Theorist, the equation $n = n + 1$ (which is equivalent to $n + 1 = n$) eventually leads to the contradictory equation 0 = 1. Theorist does not stop you from typing something like this; it depends upon you, the more intelligent human, to have the good judgment to determine that there is a mistake somewhere. (There are more complicated situations where only the human can judge what is a contradiction and what is insight; an equation such as $\sin(x) = 2$ could be a contradiction or could be a solution to a problem.)

Theorist is also non-procedural. This means the equations are not executed sequentially, they are all interpreted as a state of being, a set of facts that are assumed to be true. The order of the equations is almost irrelevant.

Theorist makes equations because if you think in terms of equations, it is harder to make a mistake. This becomes clearer later, but for now, try selecting the 3 + 4 on the left hand side of 3 + 4 = 7 and Calculate again.

$$7 = 7$$ *Calculate*

This is an equation, which is true, based upon the last equation, which is also true. All Theorist algebra is done in a similar way, by generating equations that are true because they are based upon other equations that start out being true. By maintaining this framework, Theorist ensures the integrity of your derivation.

Automatic Conclusion Elimination

Select either 7 from the last equation, 7 = 7. Then, Calculate or Simplify. Theorist does not produce an answer. If you look closely, you will notice that an answer flashes briefly where an answer would go, but Theorist deletes it quickly as soon as it realizes that the answer is the same as the question. Since there is nothing new to tell you, Theorist saves space by deleting it.

Linear Algebra

Theorist can also do matrices. For instance, make a new assumption and type **(1,2;3,4)*(5,6;7,8)**. (Be careful to type commas or semicolons where appropriate—they do different things.)

$$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}$$

Now, drag to select both matrices together—it's not much different from dragging over "(12)(34)" in a word processor—and click (or choose **Expand** from the **Manipulate** menu). (Simplify is not strong enough to do this.)

$$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}=\begin{pmatrix}19 & 22\\ 43 & 50\end{pmatrix}$$ *Expand*

Theorist is doing this symbolically, not numerically. Change the 2 in the first matrix to an x. Theorist automatically changes the Expand conclusion.

$$\begin{pmatrix}1 & x\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}=\begin{pmatrix}7x+5 & 8x+6\\ 43 & 50\end{pmatrix}$$ *Expand*

Theorist considers a scalar (a non-matrix) to be the same thing as a diagonal matrix. Press [return] and type **(1,2,3;4,5,6;7,8,9)+x**.

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}+x$$

Drag from the beginning of the expression over to the end (or double-click on the plus sign) to select it, then Expand it.

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}+x=\begin{pmatrix}x+1 & 2 & 3\\ 4 & x+5 & 6\\ 7 & 8 & x+9\end{pmatrix}$$ *Expand*

It's as if the x turned into x times the identity matrix. You do not have to make up an identity matrix; Theorist simply does the right thing for situations such as this.

MANIPULATING EQUATIONS

This chapter describes how to perform five basic manipulations:

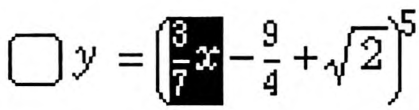
- Commute
- Isolate
- Move Over
- Expand
- Collect

In this tutorial you learn the nuts and bolts of equation manipulation in Theorist.

In a new notebook, type: **y=((3/7)*x-(9/4)+sqrt(2))^5** and choose **Clarify** from the **Notebook** menu to remove the unnecessary parentheses.

$$\square\; y = \left(\frac{3}{7}x - \frac{9}{4} + \sqrt{2}\right)^5$$

Theorist lets you do a great deal of the mundane manipulation by simply clicking and dragging. For example: in the above proposition, select the $\frac{3}{7}x$ so it looks like this:

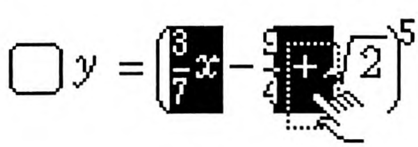


Do not be alarmed if and when Theorist changes parentheses to brackets and braces. This has no effect on how the equation is manipulated; it only affects the screen display.

Press and hold the ⌘ key (Macintosh) or Ctrl key (Windows). Notice how as you move the cursor over the selection, the cursor changes to a hand. With the ⌘ or Ctrl key still held down, click down and drag the selection around the screen. (Once the arrow pointer turns into the hand, you can let go of the key.) If you drag its outline along the equation, certain elements of the equation highlight, indicating that you can release the mouse button and drop your selection in there, but don't do it yet.

Commute

Drag your selection right, over the addition sign. When it highlights,



...release the mouse button. Theorist moves the $\frac{3}{7}x$ in between the $-\frac{9}{4}$ and the $\sqrt{2}$ so you get:

Note how Theorist removes the unnecessary brackets after the manipulation.

$$\triangle\; y = \left(-\frac{9}{4} + \frac{3}{7}x + \sqrt{2}\right)^5 \qquad \textit{Commute}$$

You can commute anything in a sum in this way. Try it with any of the three expressions added together.

Note: select a negated expression by clicking on its minus sign

You can also commute factors in a product. Try doing it on the $\frac{3}{7}x$ expression. Hold ⌘ or Ctrl and drag the x to the left, like so:

Rearranged, it becomes:

$$\triangle\; y = \left(-\frac{9}{4} + x\frac{3}{7} + \sqrt{2}\right)^5$$

Also try selecting the whole sum of three expressions and choose **Manipulate▸Other▸Commute**. This turns them all backward.

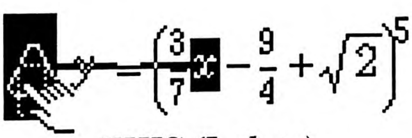
Exercise

Return the equation to its original form.

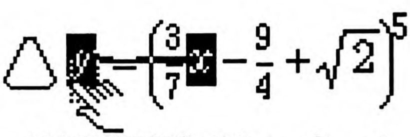
$$\triangle\; y = \left(\frac{3}{7}x - \frac{9}{4} + \sqrt{2}\right)^5$$

Isolate

The equation you have defines a relationship between two variables. There are many equivalent ways to see that relationship. One way is with x as a function of y. Select x and with ⌘ or Ctrl pressed, drag it all the way past y over the proposition icon. When the icon highlights, a line goes across the equation from the icon to the x, indicating you can drop the x to isolate it.



THIS (Isolate)



NOT THIS (Move Over)

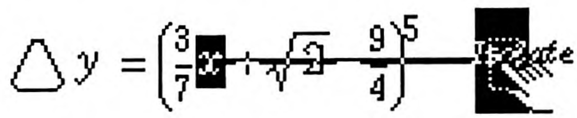
Go ahead and Isolate, which results in:

$$\triangle\; x = \frac{7}{3}\left(y^{\frac{1}{5}} - \sqrt{2} + \frac{9}{4}\right) \qquad \textit{Isolate}$$

Now use the same process to isolate the y and get back to what you had before you started experimenting with isolating.

$$\triangle\; y = \left(\frac{3}{7}x + \sqrt{2} - \frac{9}{4}\right)^5 \qquad \textit{Isolate}$$

Next, try dragging the x all the way to the right instead of left,



...which is another way of isolating x.

$$\triangle\; \frac{7}{3}\left(y^{\frac{1}{5}} - \sqrt{2} + \frac{9}{4}\right) = x \qquad \textit{Isolate}$$

Now try isolating some of the numbers. You can get some weird logarithms this way, such as:

$$\triangle\; 9 = 4\left(\frac{3}{7}x - y^{\frac{1}{5}} + \sqrt{2}\right) \qquad \textit{Isolate}$$

$$\triangle\; 5 = \frac{1}{\log_y\left(\frac{3}{7}x + \sqrt{2} - \frac{9}{4}\right)} \qquad \textit{Isolate}$$

You're in for a hard time if you try to isolate y from $\log_y$. In this case, first isolate log from the argument.

Next, isolate y to get your equation back to its original form once again.

$$\triangle\; y = \left(\frac{3}{7}x + \sqrt{2} - \frac{9}{4}\right)^5 \qquad \textit{Isolate}$$

Move Over

There is one last manipulation in this series that is somewhat different.

Using the isolate manipulation, try to get $0 = \left(\frac{3}{7}x + \sqrt{2} - \frac{9}{4}\right)^5 - y$.
You can't! The only way is by taking our original equation (way at the top of the notebook):

$$\square\; y = \left(\frac{3}{7}x - \frac{9}{4} + \sqrt{2}\right)^5$$

...and moving the y to the other side of the equal sign,

$$\square\; y = \left(\frac{3}{7}x - \frac{9}{4} + \sqrt{2}\right)^5$$

Notice the new outlining level.

...so you get:

$$\triangle\; 0 = \left(\frac{3}{7}x + \sqrt{2} - \frac{9}{4}\right)^5 - y \qquad \textit{MoveOver}$$

The palette and menu equivalents for the Isolate and Move Over manipulations are similar—but not identical—to the drag manipulations. See the Theorist Reference Manual for details.

Expand

Press return to make a new assumption and type **(x+y/z)^5** and select the expression.

$$\square \left(x + \frac{y}{z}\right)^5$$

Click (or choose **Expand** from the **Manipulate** menu), which multiplies out the terms and yields a longer expression.

$$\triangle \left(x + \frac{y}{z}\right)^5 = \frac{y^5}{z^5} + 5\frac{x^4 y}{z} + 5\frac{x y^4}{z^4} + 10\frac{x^2 y^3}{z^3} + 10\frac{x^3 y^2}{z^2} + x^5 \qquad \textit{Expand}$$

Collect

In the above equation, select the first three terms to the right of the equal sign:

$$= \frac{y^5}{z^5} + 5\frac{x^4 y}{z} + 5\frac{x y^4}{z^4} +$$

Click (or choose **Collect** from the **Manipulate** menu), which collects factors common to all terms so they are multiplied onto the side:

$$\triangle \left(x + \frac{y}{z}\right)^5 = \frac{\left(y^4 + 5x^4 z^4 + 5xy^3 z\right)y}{z^5} + 10\frac{x^2 y^3}{z^3} + 10\frac{x^3 y^2}{z^2} + x^5 \qquad \textit{Collect}$$

There are many other manipulations at your disposal, as you can see by looking at the **Manipulate** menu and its submenus. These are explored in later tutorials and explained in the Theorist Reference Manual.

You undoubtedly noticed after each manipulation Theorist appended the completed manipulation name to each conclusion. This is to show you the manipulation steps you took in your notebook's derivation(s). To hide these step names, pull down the **Notebook** menu and from its **Preferences** submenu choose **Show Steps**. Many of the screen shots in the Theorist Learning Guide and Reference Manual do not show these steps (due to space considerations), but they may be a useful learning tool for you while you do tutorial derivations.

EDITING EXPRESSIONS AND COMMENTS

Theorist utilizes two different equation editors. In assumptions and expressions you are entering normal expressions which you can manipulate or Theorist can use. In comments however, you are using the free form equation editor to enter expressions which are useful only for documenting your notebook; Theorist can not work with free form equations. Free form expressions are more decorative than functional because they can not be manipulated or otherwise used by Theorist.

The Theorist equation editor puts restrictions on what can be entered because it must understand the expressions used for the mathematics to be performed. The equation editor will not let you construct nonsense, such as "x +" (addition without a second term). Many keystrokes have special meanings to the equation editor; for example, **$** makes an integral enclosure instead of a dollar sign.

The free form editor for comments is a mixture of equation editor and word processor. You can type any combination of characters anywhere and it inserts them, just like a word processor. If you type **$**, you will get a dollar sign inserted into your text.

If you want to build real equations in a comment, use the palette or its equivalent command keystrokes (listed in Appendix A of the Reference Manual). However, just because they look real does not mean Theorist can work with them.

The free form editor allows you to change the font, size and color of text on a character by character basis, and it allows you to insert pictures.

The Theorist equation editor does not use much in the way of special formatting when displaying expressions. These expressions are always black and their limited font specifications are dictated by the Notebook Font. If you enter a new name, Theorist will later ask you what properties the name has.

To see an example of an expression with the equation editor, press [return] and type **$a** [space] **x^2**.

$$\square \int ax^2$$

Since you typed into an assumption, the equation editor turned the dollar sign into an integral, multiplied for the space, and added an exponent for the ^ caret. Now press [enter] on the numeric keypad and type the same thing, **$a** [space] **x^2**.

$a x^2

Since you typed into a comment, the free form editor just accepted what you typed literally. You can change the font, size, and color of these characters and it does not mean a thing to Theorist.

Simple Mathematics

ADDITION AND SUBTRACTION

Theorist easily performs everyday mathematical calculations. Using Theorist for these problems offers advantages over simple pocket calculators. Intermediate results may be calculated and shown, and all of your work is automatically saved in your electronic notebook. Theorist does not have to convert fractions into decimals. You may easily calculate any mix of whole numbers, decimals and fractions. Even if your interests lie in higher mathematics, you may find the basic mathematics tutorials useful to acquaint you with Theorist.

Using Theorist to add and subtract is easy. Simply type in your expression, select it, and use Theorist's Calculate manipulation. For example, type [return] **104+22+3-5**. If you prefer, you can select plus and minus from the palette instead of typing them. From the $f(x)$ palette, choosing the a+b button from the a+b pop-up palette gives you a plus sign and choosing a−b (from the same pop-up) gives you a minus sign.

Now select what you just entered by dragging (or double clicking on a plus or minus sign). Click the [calculator] palette button (or choose **Calculate** from the **Manipulate** menu).

$$\square\ 104+22+3-5$$

$$\triangle\ 104+22+3-5=124 \quad \textit{Calculate}$$

You can have Theorist give you an immediate result by using Calculate as we just did in the previous example. Or you may want to compute intermediate results, as in the following example.

You may easily include fractions and decimals in numerical expressions. Type [return] **2+1/2** [esc] **+1/4** [esc] **+0.25**. This produces the following expression:

$$\square\ 2+\frac{1}{2}+\frac{1}{4}+0.25$$

Now let's determine the sum of the two fractions with the Simplify command. Drag to select the fractions as shown:

$$\square\ 2+\frac{1}{2}+\frac{1}{4}+0.25$$

Click the palette button (or choose **Simplify** from the **Manipulate** menu). This combines the two fractions into a single fraction with a common denominator.

$$\triangle\ 2+\frac{1}{2}+\frac{1}{4}+0.25=2+\frac{3}{4}+0.25 \qquad \textit{Simplify}$$

Calculate the 3/4 term.

$$\triangle\ 2+\frac{1}{2}+\frac{1}{4}+0.25=2+0.75+0.25 \qquad \textit{Calculate}$$

Finally, select the right side of the equation and Calculate to get a single numeric result:

$$\triangle\ 2+\frac{1}{2}+\frac{1}{4}+0.25=3 \qquad \textit{Calculate}$$

In the example above, Calculate was used to change a fraction into a decimal. Theorist also provides an UnCalculate command that converts decimal numbers to their simplified fractional equivalents, if possible. Let's find out how many fourths equal 5.75. Type [return] **5.75**, select it, and choose the UnCalculate button from the pop-up sub-palette (or choose **UnCalculate** from the **Manipulate▸Other** submenu):

$$\square\ 5.75$$

$$\triangle\ 5.75=\frac{23}{4} \qquad \textit{UnCalculate}$$

MULTIPLICATION AND DIVISION

Theorist lets you enter expressions involving multiplication and division from the keyboard or from the palette. From the keyboard, use the asterisk * or [space] for multiplication and the slash / for division. From the $f(x)$ functions palette's a+b pop-up sub-palette of operators, choose a·b and a/b for multiplication and division, respectively.

Multiplication is represented in different ways at different times, depending on context. This is true in mathematics textbooks as well as in Theorist. For example, type [return] **5*5** and Theorist displays:

$$\square\ 5\cdot 5$$

Whereas typing [return] **5*(3+4)** displays:

$$\square\ 5(3+4)$$

Multiplication and division problems are simplified and calculated in the same way as addition and subtraction problems. Select the portion of the expression you wish to calculate or simplify, then choose the desired command from the **Manipulate** menu or functions palette.

When dividing, Theorist automatically represents the divisor and dividend as a fraction. To enter the expression,

$$12 \div \frac{3}{5}$$

...press [return] and type **12/3/5**. Theorist represents the expression as:

$$\square\ \frac{12}{\frac{3}{5}}$$

This of course is equivalent to the original expression, 12 divided by 3/5. If you only want the answer to the problem, you may quickly get it by selecting the expression and using Calculate. If you prefer to work the problem out manually, however, Theorist allows you that option as well.

To work the problem out manually, as you would using paper and pencil, first multiply the numerator and denominator by 5. Select the

It is not necessary to use a multipli-cation operator because Theorist assumes that is what you want.

whole fraction then click (or choose **Apply** from the **Manipulate** menu).

$$\frac{12}{\frac{3}{5}} = \frac{\blacksquare \cdot 12}{\blacksquare \frac{3}{5}}$$

At this point, the Apply manipulation is still in effect. Type the number **5**, which appears in both places because of the multiple selection.

$$\frac{12}{\frac{3}{5}} = \frac{5 \cdot 12}{5 \frac{3}{5}}$$

Now select the new numerator (12 times 5), which ends the Apply manipulation. (Hence the beep. The step name appears after the conclusion.) Click the palette button (or choose **Simplify** from the **Manipulate** menu). Select and Simplify the denominator, and the entire fraction, which gives you a final answer of 20.

$$\frac{12}{\frac{3}{5}} = \frac{5 \cdot 12}{5 \frac{3}{5}} \qquad \textit{Apply}$$

$$\frac{12}{\frac{3}{5}} = \frac{60}{5 \frac{3}{5}} \qquad \textit{Simplify}$$

$$\frac{12}{\frac{3}{5}} = \frac{60}{3} \qquad \textit{Simplify}$$

$$\frac{12}{\frac{3}{5}} = 20 \qquad \textit{Simplify}$$

This is a lot of work which is better accomplished in one fell swoop with Theorist, but it shows you the individual control Theorist gives you.

FRACTIONS

Adding fractions by finding a common denominator may be performed automatically by selecting the fractions and using the Collect or Simplify Manipulations:

Never use commas to enter numbers in Theorist.

$$\frac{1}{6000}+\frac{1}{8888}=\frac{1861}{6666000}$$

Prime Factors

Alternately we may work this problem out manually by using Theorist to find the prime factors of 6000 and 8888. Double-click 6000 to select it and hold [shift] as you double-click 8888 so you have a multiple selection. Click (or choose **Factor** from the **Manipulate** menu).

$$\frac{1}{6000}+\frac{1}{8888}=\frac{1}{2^4\cdot 3\cdot 5^3}+\frac{1}{2^3\cdot 11\cdot 101}$$

Factor Factor

Using Parentheses and Escape

When you enter a division problem, you will sometimes have to use the Escape key or parentheses to convey your intentions to Theorist. Consider the difference between these two expressions:

$$\frac{1}{3}+4 \text{ and } \frac{1}{3+4}$$

To enter the first, type **1/3** [esc] **+4**. Pressing the Escape key makes it clear to Theorist that you are finished entering the denominator. To enter a denominator of "3+4", leave out the Escape key. If you are ever unsure as to what to do, use parentheses, as in **1/(3+4)**, which produces the following.

$$\frac{1}{(3+4)}$$

You can always remove unnecessary parentheses later with the **Clarify** command.

Names

NAME DECLARATIONS

In Theorist, all constants, variables, and functions are assigned particular names made of upper- and lower-case characters. This tutorial provides a brief description of:

- Name declarations
- Name behaviors
- Changing a name
- Getting information about a name
- Name classes

Earlier in the manual, we mentioned there is something special about the top comment, Declarations. Double-click on the comment icon to expose what is beneath it:

For now, ignore the collapsed comment bundles.

Trigonometry

Hyperbolic

Logarithms & Powers

Standard Rules

A [Constant] named i behaves as [i: Square root of -1].

A [D-Linear Operator] named d behaves as [d: differential operator].

A [Variable] named k behaves as [defined by user].

A [Constant] named n behaves as [defined by user].

A [Constant] named c behaves as [defined by user].

A [Constant] named b behaves as [defined by user].

A [Constant] named a behaves as [defined by user].

A [Variable] named z behaves as [defined by user].

A [Variable] named y behaves as [defined by user].

A [Variable] named x behaves as [defined by user].

Each item with a flag icon is a name declaration. When you create a new notebook (based on the new notebook file), it has these declarations.

Each part of a declaration specifies some facet of operation for that name. The two pop-up menus are used to specify the class and the behavior of the name indicated in the middle of the declaration.

Behavior

The behavior can be "defined by user" or it can be one of dozens of special behaviors. For example, type the name "Fred" in a new assumption. Choose **Clarify** from the **Notebook** menu, and when the dialog comes up, specify user-defined. Afterwards, your name declaration for Fred appears under the Declarations comment. (This is where new name declarations go.)

A [Variable ▾] named Fred behaves as [defined by user ▾].

Using the behavior pop-up menu, which currently says **defined by user**, change Fred's behavior to **i: Square root of −1**, which is under the **Constants** submenu.

A [Constant ▾] named Fred behaves as [i: Square root of -1 ▾].

Now close the Declarations (double-click the comment icon again) and work on the "Fred" expression. Click to place the cursor after the "d" and type **^2**:

$$\text{Fred}^2$$

Select the expression and click either the or palette buttons to Calculate or Simplify it.

$$\text{Fred}^2 = -1$$

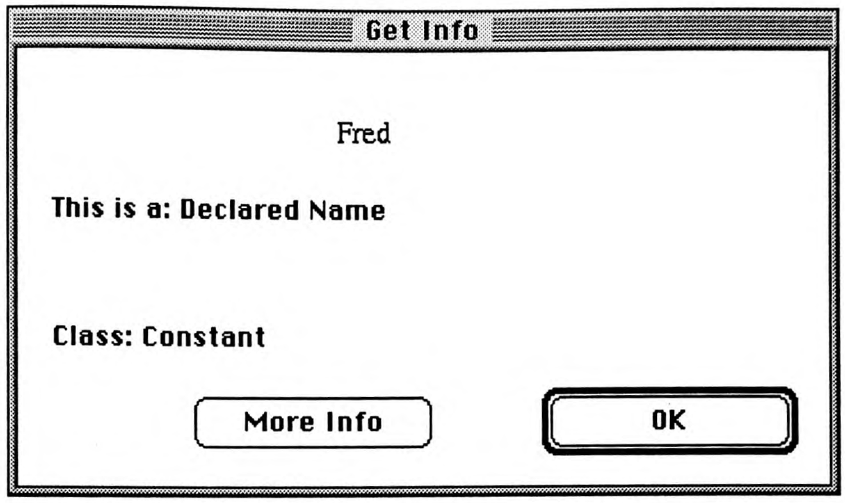
This is not just an alias, as with some computer languages. As far as Theorist is concerned, "Fred" *is* the name of the imaginary number.

Type [return] **sqrt(-1)**, select the resulting expression, and evaluate it as you did the other one:

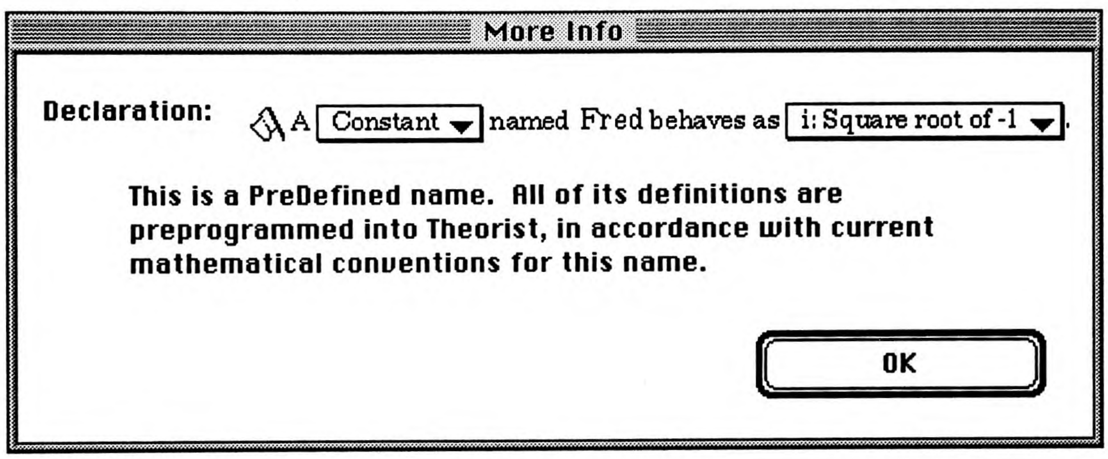
$$\sqrt{-1} = \text{Fred}$$

Now, if you want, you can have the name *i* refer to something completely different. You can use it to represent the current in a wire, for instance.

Select any occurrence of Fred in any assumption or conclusion and choose **Get Info** from the **Notebook** menu.

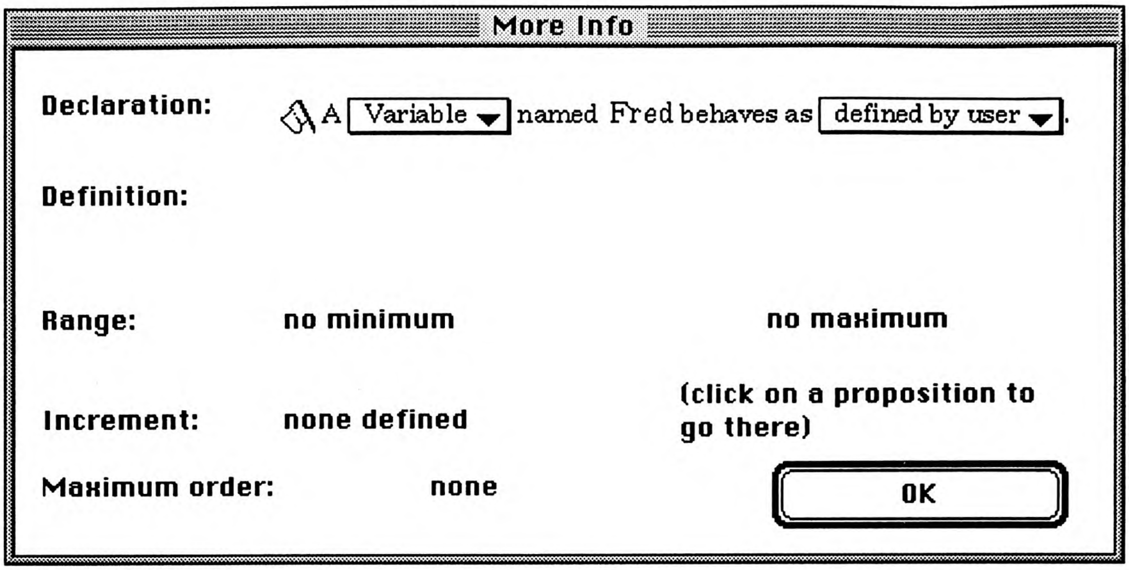


Click on **More Info** in the dialog.



Theorist tells us it is a predefined name, and that's that. Click **OK**.

If Fred were user defined rather than predefined, you would have some more options. Let's take care of that. Expose the Declarations bundle and change Fred's behavior to **defined by user** then Collapse the bundle. Select any Fred in any assumption or conclusion and choose **Get Info**, as before. Click **More Info** in the dialog.



Click on the declaration part of the dialog, and Theorist leaves the dialog and goes to the Declarations part of the notebook and selects the declaration for Fred.

A Variable named Fred behaves as defined by user

Change the name Fred to Sally. Watch Fred change to Sally everywhere.

$\square\ \text{Sally}^2$

$\triangle\ \text{Sally}^2 = -1$ *Calculate*

$\square\ \sqrt{-1}$

$\triangle\ \sqrt{-1} = \text{Sally}$ *Simplify*

Legitimate names may consist of any upper and/or lower case letters and any Greek letters and other symbols from the Greek sub-palette. They may not contain digits, underscores, or other special characters. See the *Theorist Reference Manual* for more details.

Name recognition is case-sensitive. Names can be multiple characters; single-letter names are displayed in italics when the Notebook Font italics option is on.

Create a new assumption and type **a=Sin(x)+SIN(y)+sIn*(z)**, paying particular attention to upper and lower case.

Clarify the notebook. Note that Theorist asks you to declare Sin, SIN, and sIn, because it recognizes none. (Make them whatever you want.) Theorist does not confuse them with the predefined sine function, “sin” (all lower case).

Class

Each name belongs to one of five classes:

- Constant
- Variable
- M-Linear Operator (matrices)
- D-Linear Operator (derivatives and partial derivatives)
- Function

A name's class determines some of its properties.

Create a new assumption and type **d*(A*B)** so you have the following.

$$\square\ d(AB)$$

The name *d* is predefined.

Clarify and declare *A* as a constant and *B* as a variable. Simplifying the expression gives you:

$$\triangle\ d(AB) = A\,dB \quad \textit{Simplify}$$

Using the method described previously (Get Info/More Info), change *A* to a variable. Simplify the original expression again.

$$\triangle\ d(AB) = B\,dA + A\,dB \quad \textit{Simplify}$$

Make a new assumption and type **(A+B)^2** so you get:

$$\square\ (A+B)^2$$

Expand it to get:

$$\triangle\ (A+B)^2 = A^2 + B^2 + 2AB \quad \textit{Expand}$$

Change the name declarations for both *A* and *B* to M-Linear operators, which causes them to behave like matrices. Expand $(A + B)^2$ again.

$$\triangle\ (A+B)^2 = A^2 + BA + B^2 + AB \quad \textit{Expand}$$

Because M-Linear operators do not commute, they do not combine.

Change *B*'s class to function. Notice how your expressions change. Functions tend to apply themselves to whatever they find nearby.

Make a new assumption and type **d(d[x*y])** so you get:

$$\square\ d(d[xy])$$

The d operator takes derivatives. Its class is D-Linear operator. Simplify the expression and notice the resulting expression has parentheses for grouping:

$$\triangle\ d(d[xy]) = y\,d\,d\,x + 2(d\,x)d\,y + x\,d\,d\,y \qquad \text{Simplify}$$

D-Linear operators are not necessarily associative. For instance, the expression *(dx)dy* is not the same as *dxdy*. In the latter case, the first d applies to everything to its right; in the former case, the first d is applied only to x.

Graphing

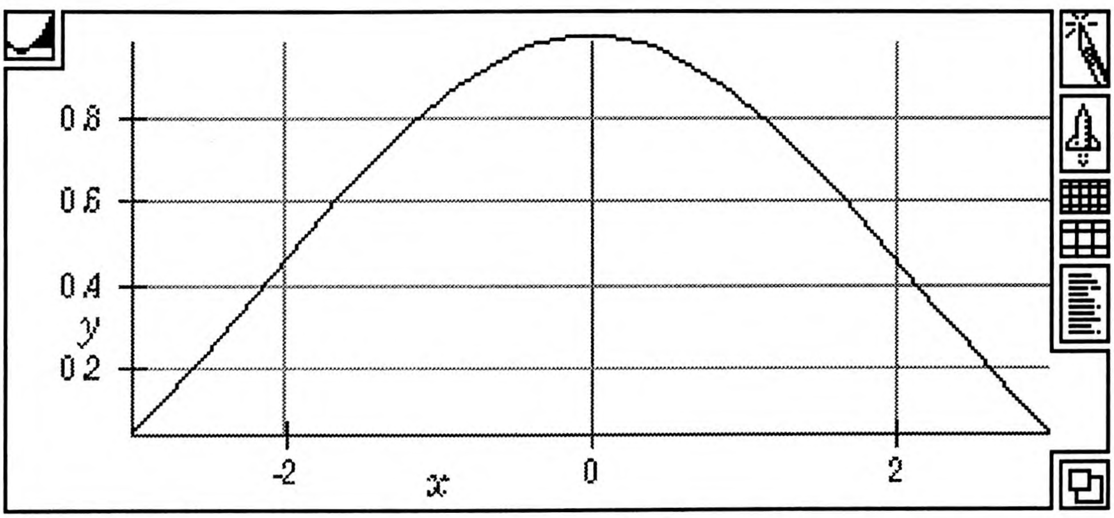
TWO-DIMENSIONAL GRAPHS

This tutorial describes the basic tools you can use to interact with two-dimensional graphs, including:

- The hand cursor, for manipulating the graph directly
- The Rocket, for zooming out and in on a graph
- The Knife, for selecting a portion of a graph
- Two buttons for increasing and decreasing plot resolution
- The size box, for resizing the graph window

In a new notebook, type **y=sin(x)/x**.

With the cursor still in the equation, click the palette button (or pull down the **Graph** menu and from its **y = f(x)** submenu, choose **Linear**).



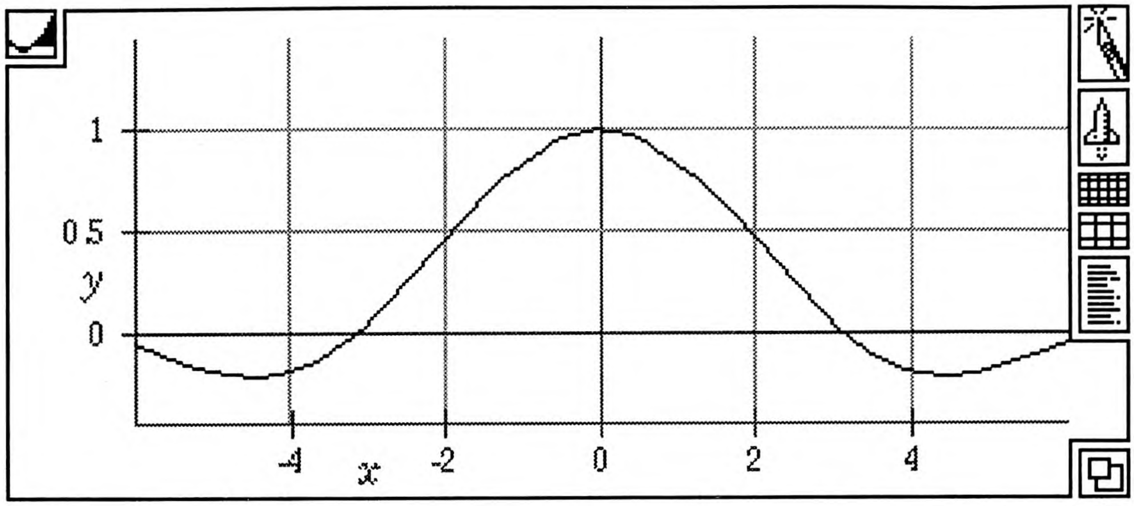

Hand Cursor

This only shows a small part of a very interesting graph. When you move the cursor over the graph surface, it turns into a hand. You can scroll the graph with the hand by dragging it. As you do, the axis labels change to indicate what part of the graph is visible.

By default, Theorist scrolls the graph smoothly as you drag it around. How smooth it is depends on how fast your computer is. If you are using a slower machine it may be "jumpy." If this is the case, choose **Graph▸Preferences...** and change the Smooth Scrolling resolution to a lower number.


Rocketship Button

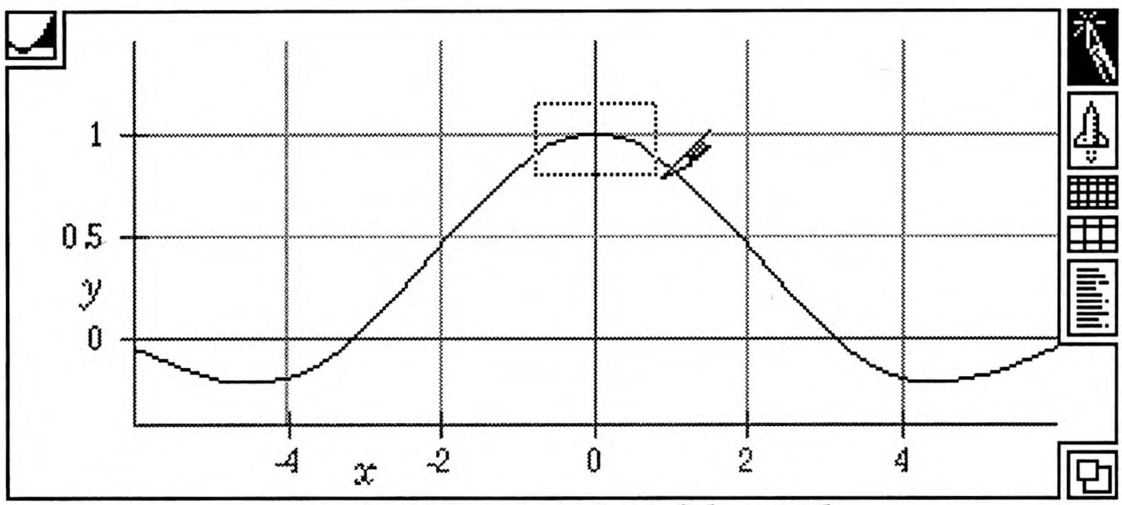
If you get frustrated crawling along a graph and want to get a more global view of it, simply click on the Rocketship icon in the upper-right section of the graph window. It zooms your graph out by about a factor of two in each dimension.



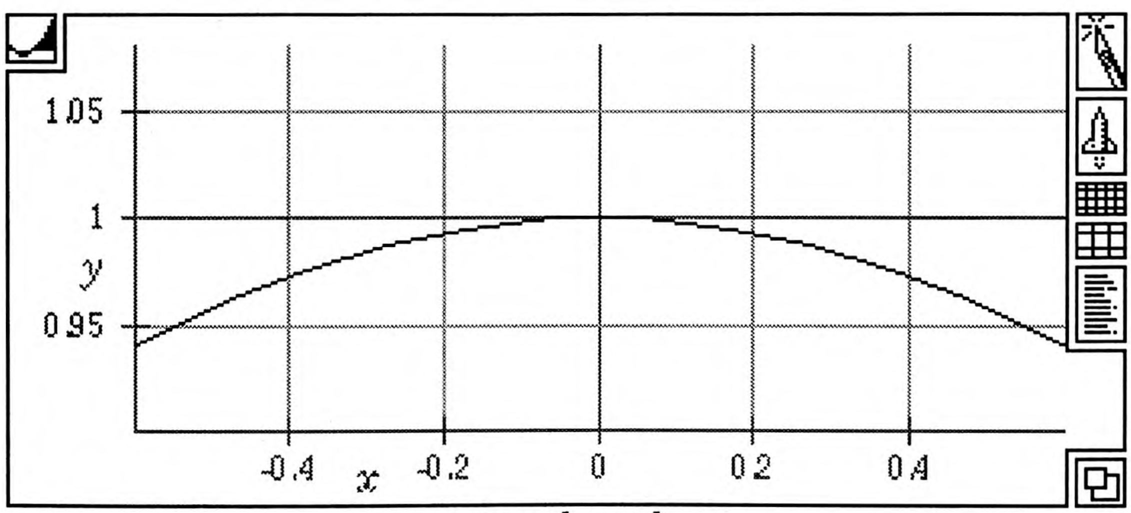
Knife Button

Tip: You can zoom in on a graph by holding the option key (Macintosh) or alt and Ctrl keys (Windows) as you click the Rocketship.

To zoom in on a specific section of the graph and see more detail, click on the knife icon. Your cursor turns into a knife, ready to slice out and enlarge a piece of the graph. Click on the graph surface itself and drag over a rectangle that you want to see closer. It is usually best to drag your rectangle so that it tightly encloses the part you want to see:



zooming in on a portion of the graph



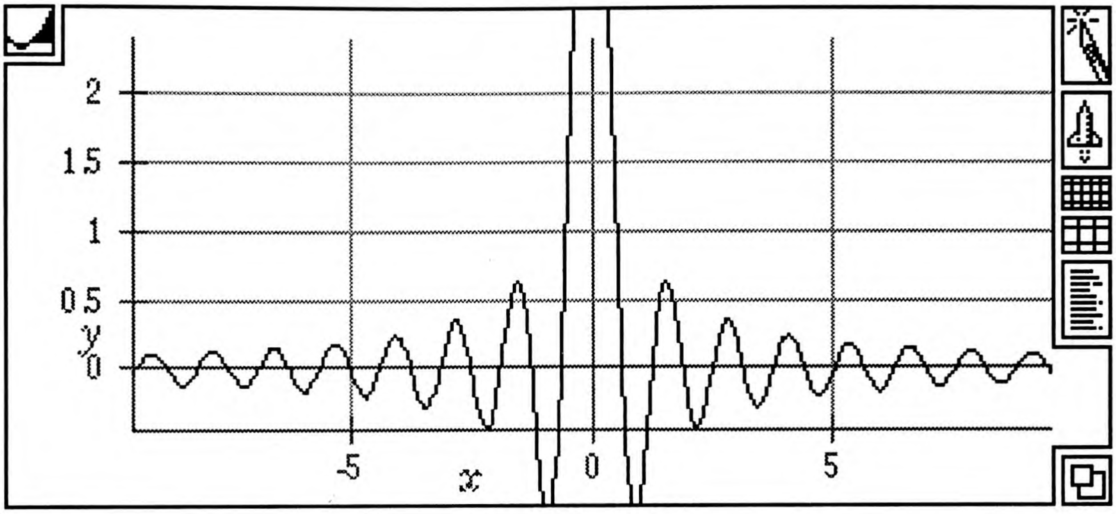
zoomed graph

More Accuracy Button

The More Accuracy button increases the number of data points for a higher resolution plot, which makes certain types of plots smoother. (This is not one of them.) As you click the button, notice that two numbers appear next to it. The left number is the current resolution, and the right number will be the new resolution when the graph is redrawn. Right now, increase the resolution a few clicks.

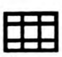
Notice the drawing speed as you type and zoom out.

Theorist graphs are interactive and always reflect current conditions. When you change the equation, the graph changes in response. For instance, select the x from sin(x) in your equation and type ***5**. Use several clicks of the Rocketship to zoom out.



Dragging Size Box

The size box in the bottom-right corner of the graph window can be used to resize the graph. Drag it outward to see a bigger graph or inward (toward the proposition icon) to see a smaller graph. Notice the graph dimensions displayed in points, inches, and millimeters. When you release the mouse button the graph is redrawn.



Less Accuracy Button

Have you noticed the slower graph drawing speed since you increased the resolution with the More Accuracy button? That is a side effect of higher resolution. Fortunately there is the Less Accuracy button, which has the opposite effect of the More Accuracy button. It decreases the number of data points for a lower resolution plot. Its own side effect is it makes certain types of graphs more chunky or jagged. This is not a problem with this particular graph. Give the Less Accuracy button a few clicks now to get back to the original resolution.

Exercise

Make a 2-D plot of $y = \sin x$ and note the numbers on the axis labels. Zoom out a couple times, then try to zoom back in by using the knife to cut out a portion in the original dimensions. See how close you can get with practice. Remember that, if necessary, you may actually drag the knife outside the graph area.

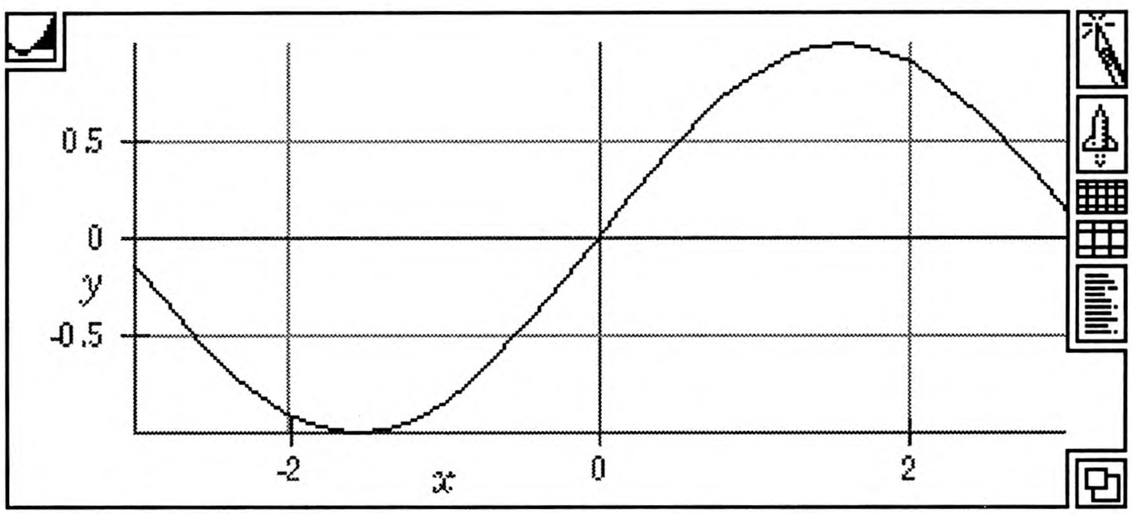
Super Knife Mode

When examining a graph it is often necessary to Zoom in on a particular region of the graph. This can become tedious and time consuming when the standard Knife tool must be used more than a few times. Super Knife mode relieves you of this tedium. To activate Super Knife mode, double click on the Knife button of a graph.

Super Knife mode has two advantages over the original Knife mode:

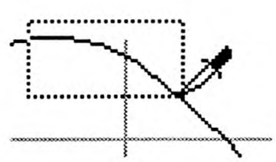
- It stays on by itself. After one slice, you do not have to click on the icon to slice again; it stays on until you click the button again to turn it off.
- As far as Undo is concerned, all of the slices combined are considered one operation. So Undo undoes *all* of your slices and returns the graph to its original (pre-Super Knife) view.

To demonstrate Super Knife, make a 2-D linear graph of some sort.



Super Knife tool

Double click on the Knife button. The cursor turns to a knife, but with a black handle rather than the white handle of the normal knife cursor. Choose a section of the curves at random and slice out a small rectangular piece along the curve.



(It usually helps to click the knife down right on or near the graph line and drag in any direction that will encompass part of the line. If on a diagonal, click down along the line, drag along its length a short way, and release. That way when the graph is blown up the curve comes in and out through the corners and you get a good view of the plot in the middle.)

With Super Knife mode, you can slice again and again in rapid succession because the knife stays active. (The mouse pointer remains the black knife instead of returning to the arrow pointer, as it would if the regular knife were used.) If you take small slices, you can get down a factor of ten for each slice. If you take larger slices, you may only get down a factor of two or three. Keep on slicing until you see the two lines splitting apart from each other. This shows you how accurate the table is in that particular region of its domain.

If you do any other editing command, it, not the slicing, is undone.

Now, before doing anything else, Undo, which returns the graph view to its previous state.

TWO-DIMENSIONAL GRAPH DETAILS

Each graph is defined by a set of statements that describe the contents of the graph. When you edit a graph using the Rocket, Knife, or Hand, you change certain values in these statements. You can also edit these statements directly. This tutorial provides a brief tour of the graph details for a two-dimensional graph, including:

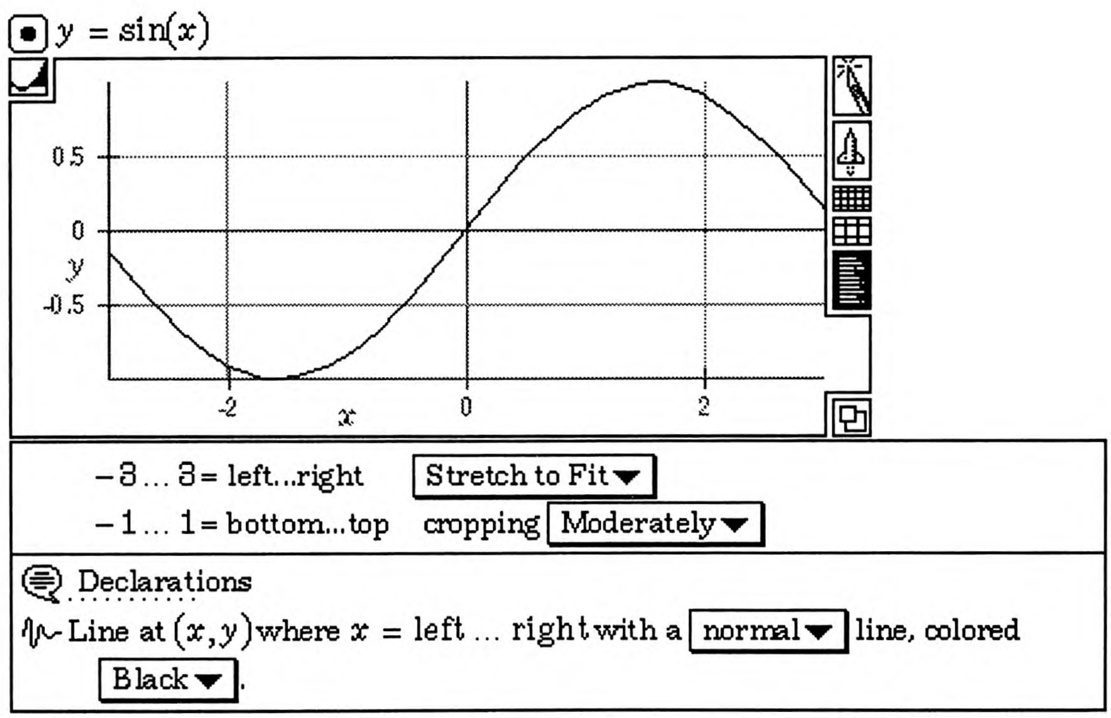
- Graph bounds
- Axes propositions
- Grid Line propositions
- Plot propositions

A Theorist graph is a space for drawing graphic objects. A plot is a graphic object drawn inside a graph. In this example, we explore a graph and its plots.

Type **y=sin(x)** into a new notebook.

Graph Details Button

Create a graph by clicking the palette button (or choosing **Graph▸ y = f(x)▸Linear**), then click on the graph details button to see details about the graph.



Locate the first two lines in the graph details:

−3 ... 3 = left...right

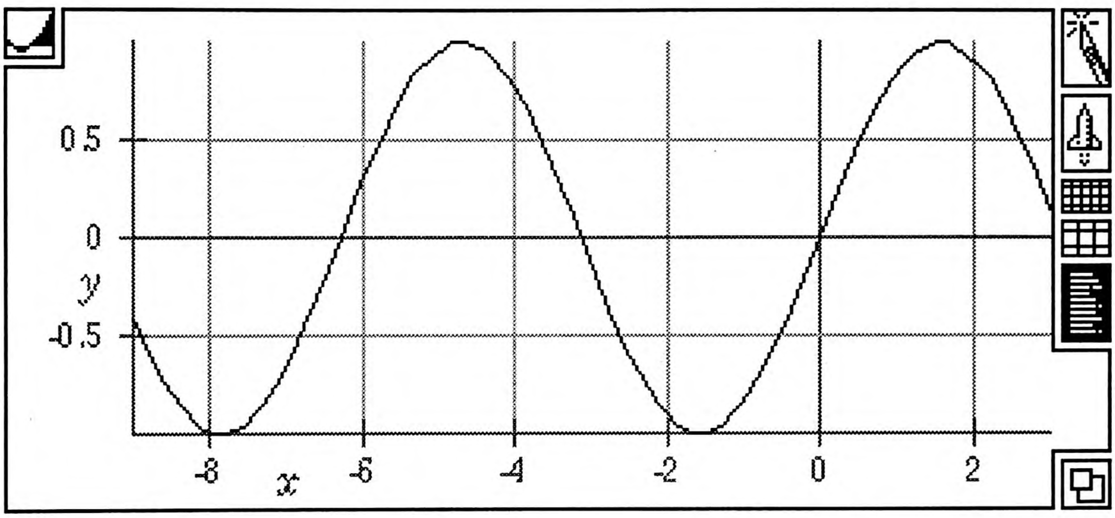
−1 ... 1 = bottom...top

These define the graph bounds, or the range of x and y values visible within the viewport. Look at the x-axis on the graph and note that it ranges from –3 to 3. (Although it is only labeled at –2, 0, and +2, you can see where –3 and +3 are).

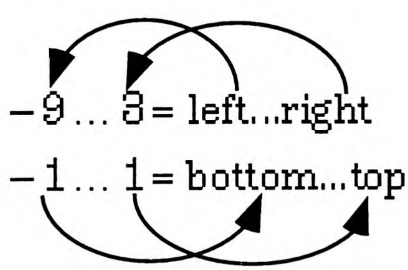
Change the –3 (which corresponds to left) to –9. You need not disturb the minus sign. The graph redraws itself, this time showing more real estate on the left side, all the way to –9.

−9 ... 3 = left...right

−1 ... 1 = bottom...top



Note that the names imply their purpose.



Similarly, you can change the other three and manipulate them as desired. Note that when you use the knife or Rocketship you are actually causing Theorist to intelligently modify these values and, subsequently, the graph.

Note the elements in the image of the graph. (See below.) There are vertical and horizontal axes, a sine wave, and crosshatched grid lines.

Each is independent and can be manipulated individually, allowing you to control the appearance of the graph.

Below the graph, in the details, you can see the line plot proposition. If you expose the declarations within the graph details (by double-clicking the comment icon) you can see other plots that allow you to control the graph. Each icon starts a new plot. There are two axis plots, one line plot, and two grid plots (one for vertical lines and one for horizontal). Manipulating the plots allows you to control what is drawn.

Declarations

Grid lines at (x, y) where y = bottom ... top for each value x = left ... right separated by 0 colored [Lilac ▾].

Grid lines at (x, y) where x = left ... right for each value y = bottom ... top separated by 0 colored [Lilac ▾].

Axis at (left, y) where y = bottom ... top labeled y on [other side ▾] colored [Lilac ▾].

Axis at (x, bottom) where x = left ... right labeled x on [this side ▾] colored [Lilac ▾].

A [Constant ▾] named left behaves as [Horizontal Minimum ▾].

A [Constant ▾] named right behaves as [Horizontal Maximum ▾].

A [Constant ▾] named bottom behaves as [Vertical Minimum ▾].

A [Constant ▾] named top behaves as [Vertical Maximum ▾].

Line at (x, y) where x = left ... right with a [normal ▾] line, colored [Black ▾].

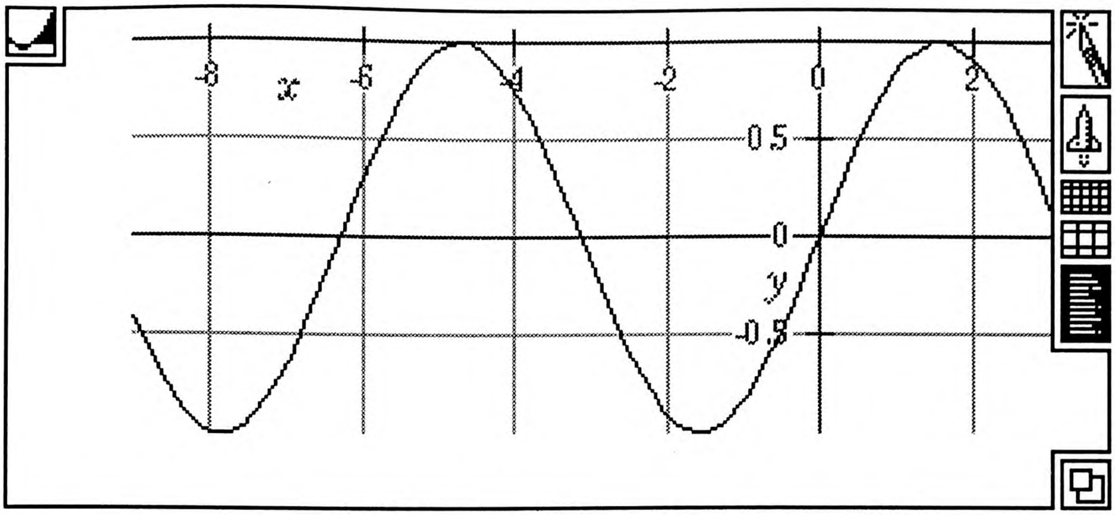
Axes

You can move the bottom horizontal axis to the top. Find the axis plot that starts out "x, bottom". Change the word "bottom" to "top". Use the same technique to change the vertical axis from the left side to the right. You can also move the vertical axis to zero (or another value if you want) by typing **0** in place of "left" or "right":

Axis Icon

Axis at (0, y)

Axis at (x, top)



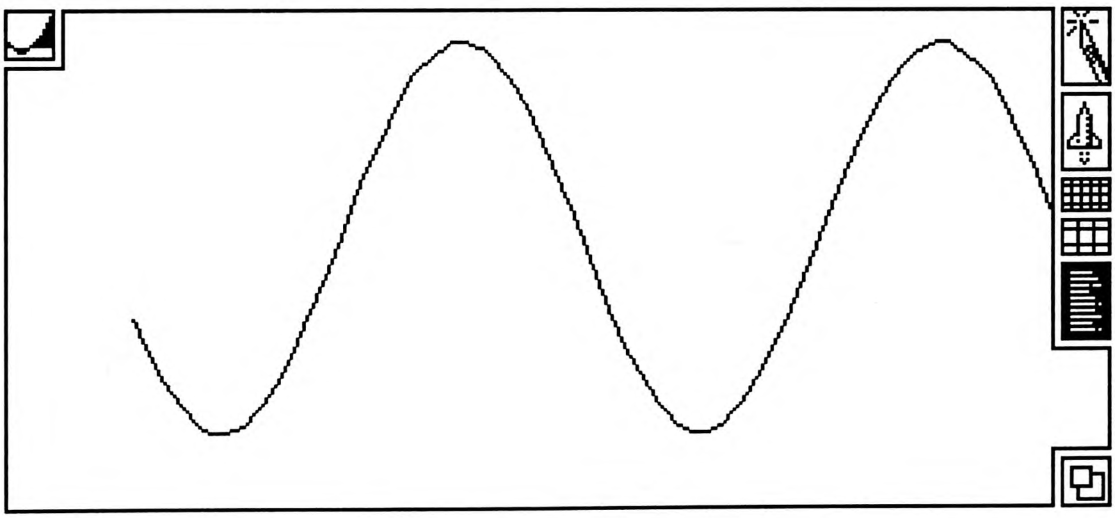
You can move the axis to any numerical value by typing in that value.

You can remove an axis by clicking on its proposition icon to select it, then pressing Delete. Do this now to both axis propositions and the graph redraws with no axes.

Grids

Grid Icon

One grid proposition defines the vertical lines and the other defines the horizontal lines. You can remove a set of grid lines by clicking on its icon to select it, then pressing Delete. Do this now to both grid propositions and the graph redraws with no grids.



Line Plots

You can delete the line plot itself. Do this now and notice the graph has nothing drawn in it. Since there is nothing left in this graph, delete it by clicking on its icon and pressing delete. Select some part of your equation and choose **Graph▸y = f(x)▸Linear** to make another graph with everything in it.

Line Plot Icon

Now we will learn in detail how a plot works. Open up the graph details and examine the line plot:

Line at (x,y) where x = left ... right with a [normal▼] line, colored [Black▼].

Line Style and Color

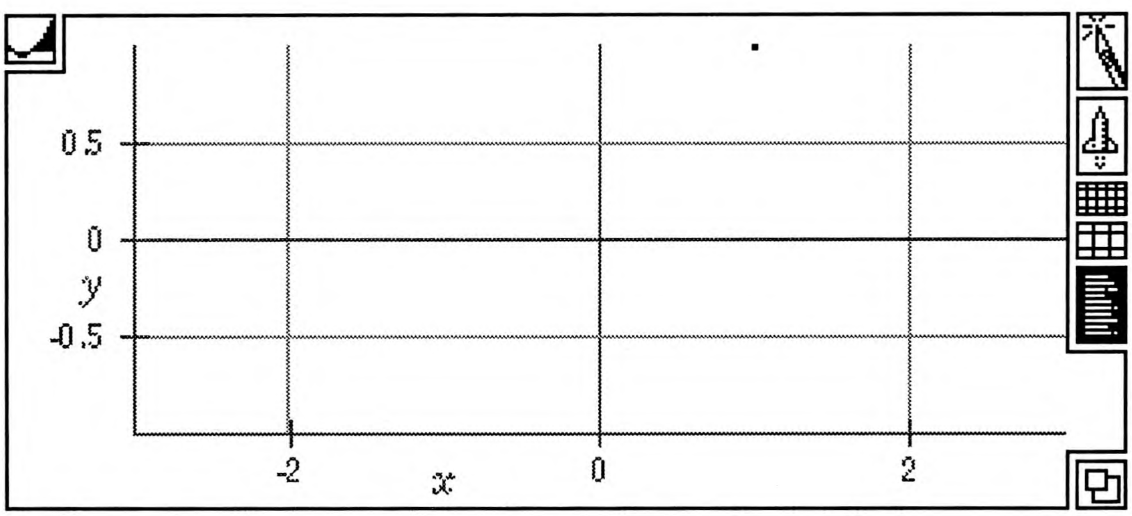
The parts which say normal and Black are the line style and line color pop-up menus, respectively. You can change the line style to any one of four options. You can change the line color to one of many, depending on your computer. These are useful if you have more than one line plot in a graph. Change it to dashed, dotted, then heavy and look at the results after each change. Leave it at heavy.

Vector

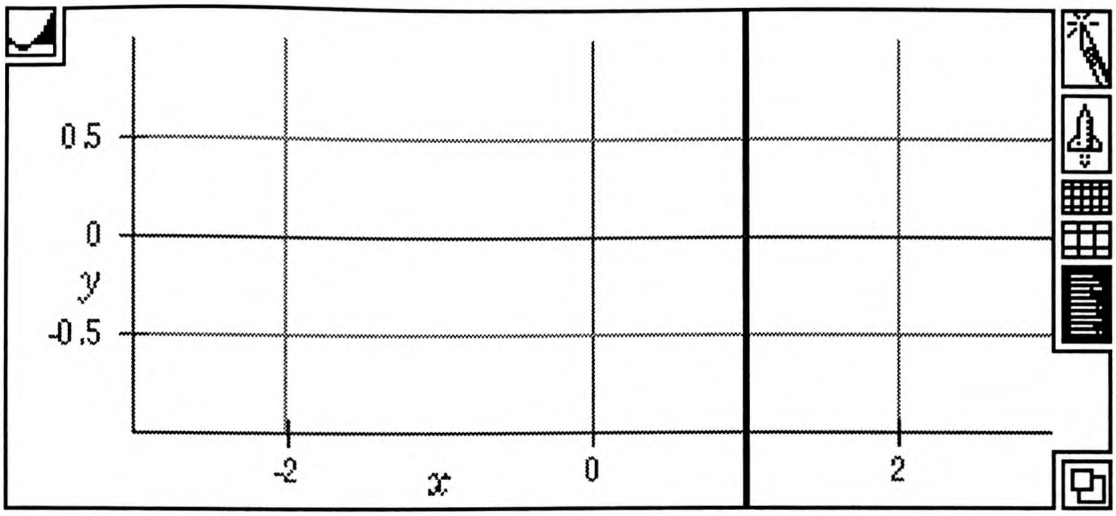
The first part, (x,y), is the location vector. This is an expression that must, as the name implies, be a 2-vector. If this were a 3-D graph, it would have to be a 3-vector. 2-vector simply means it is two expressions with a comma in between.

For education's sake, you can change this vector to something very simple. Change the vector to (1,1) by selecting the whole vector and typing in the coordinates.

Notice that your graph turns into a single dot at (1,1). As you can see, the vector is a very important part of the line plot.



Next, change the second 1 to x. The graph turns into a vertical line.



The first coordinate of this line is always 1, while the second coordinate ranges from –3 to +3, as the second part of the line plot implies. It's just that the names are confusing.

You are not limited to *x* and *y*; you can use any variables.

Theorist is designed to draw a wide variety of plots, some of which are plots of *y* as a function of *x*. Some are plots of a variable named *P* as a function of *T*. Some are even plots of *x* as a function of *y*. As such, the vertical coordinate is not necessarily *y*, and horizontal coordinate is not always *x*. You can use any variables for these purposes.

In Theorist, the rule is that the first coordinate is the horizontal one and the second the vertical. You can see this both in the order of the vector and in the order of the bounds at the top of the graph.

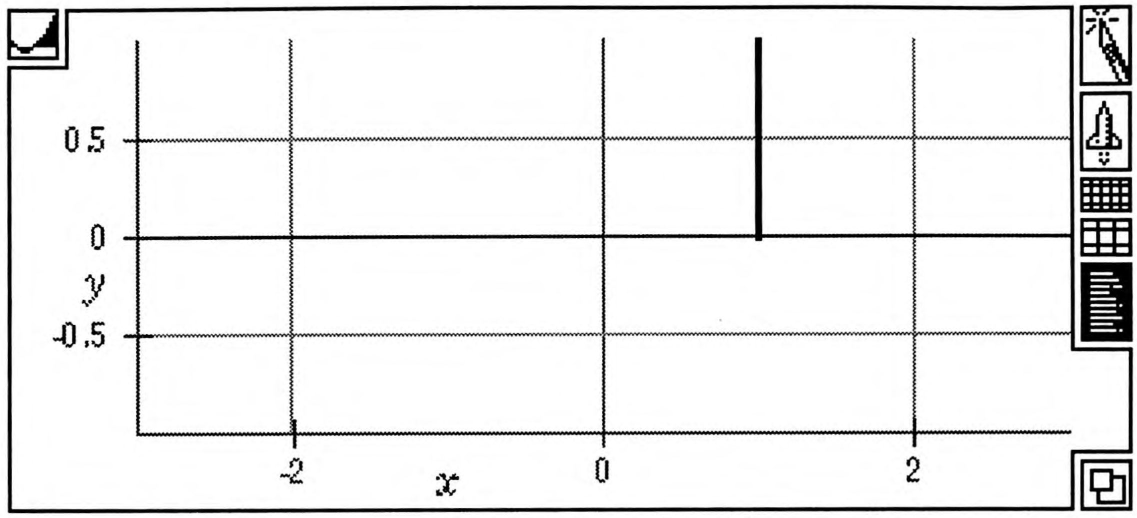
Bounds

As you can see, the second part of the line plot, "x = left...right" is starting to become significant.

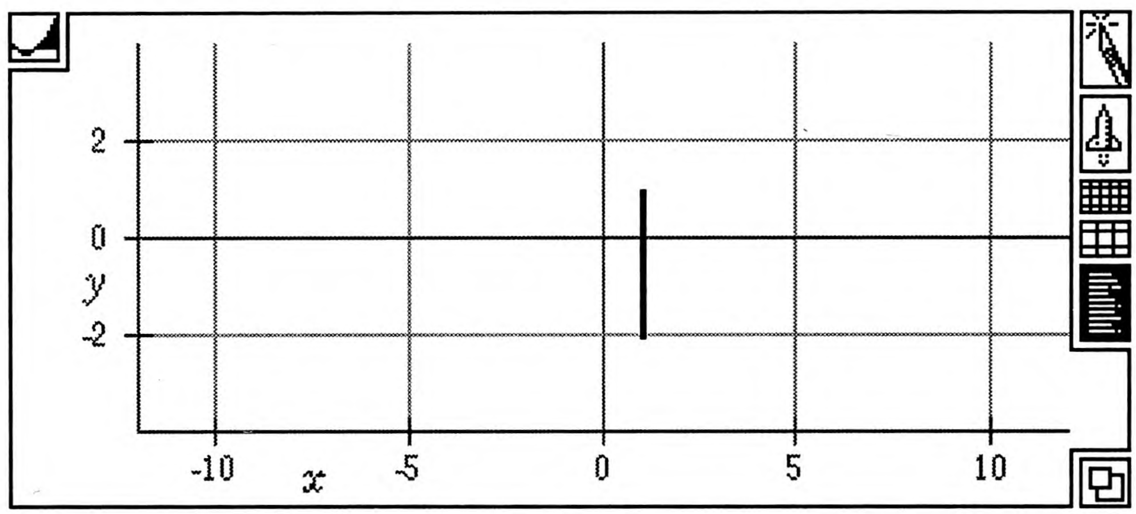
Remember we are still referring to the line plot.

The words left and right are simply names that are only allowed inside of a graph. They represent the graph bounds at the top of the graph. At present, the right represents 3 and the left –3. Change this to read "x = 0...1". (Type periods for the "..." part. If you have trouble, type **0.0** for 0 to make things perfectly clear to Theorist.)

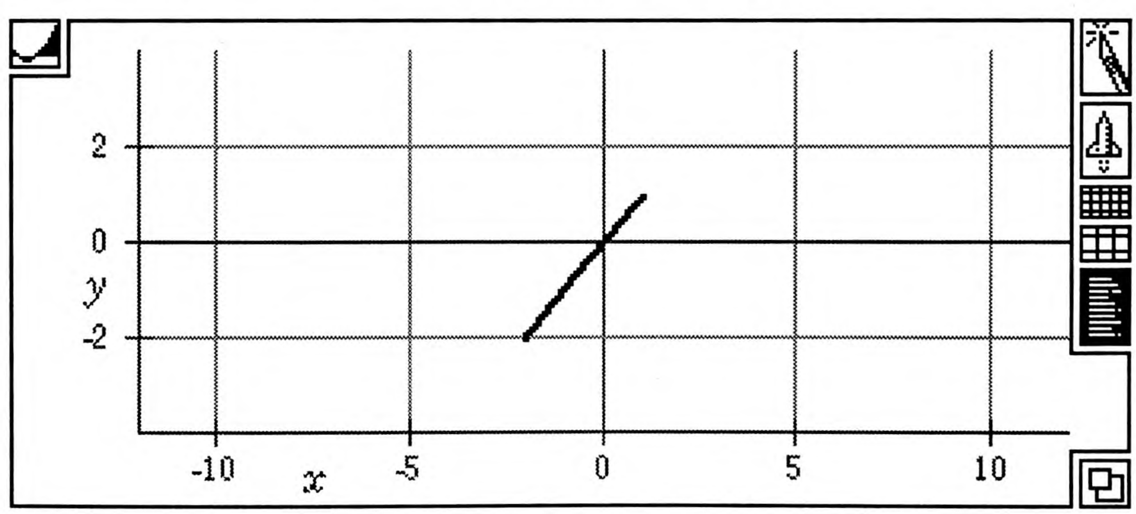
Your graph now looks like this:



Change it to "$x = -2 \dots 1$", and zoom out with the rocket a couple times to see the endpoints:



Now change the vector from $(1, x)$ to (x, x). The graph looks like this:



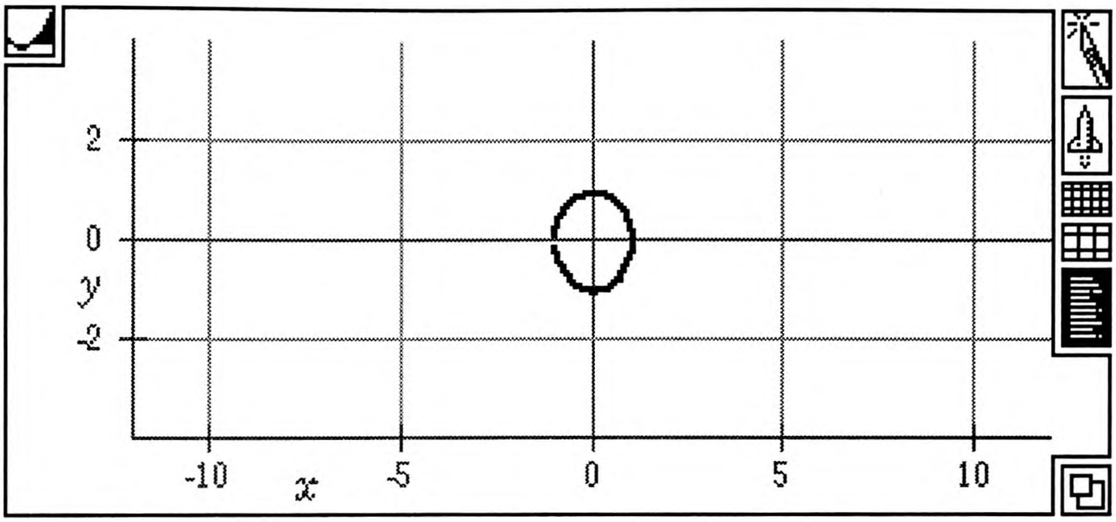
To find a point to plot, Theorist calculates the vector. To get successive points to make a line, Theorist defines x to have values ranging, in this case, from –2 to 1.

In other words, the line plot is a miniature parametric plot. Change it to the following:

Line at $(\cos[x], \sin[x])$ where $x = -3 \ldots 3$

If the range is $-\pi$ to π, the two ends meet.

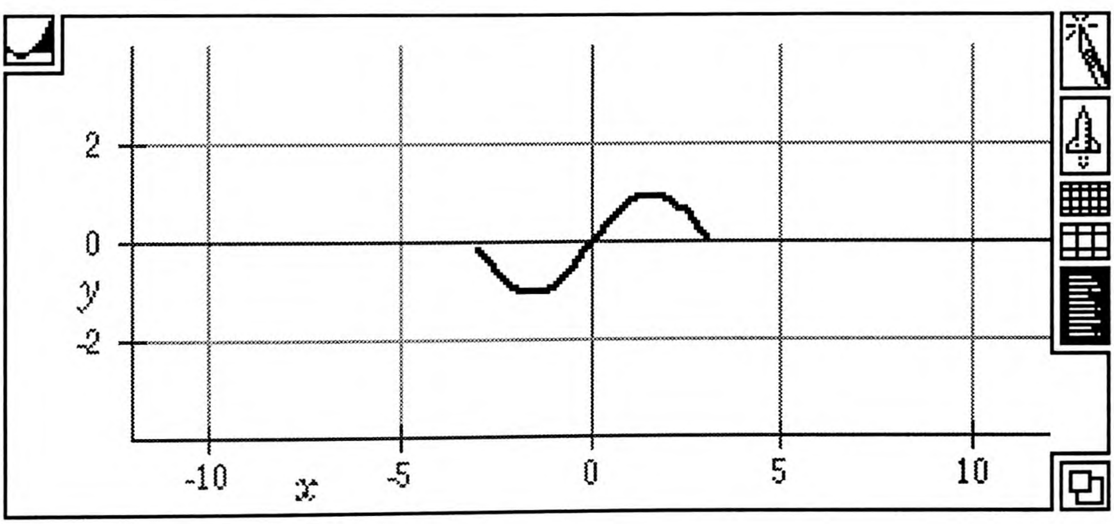
Theorist plots an incomplete circle.



Now change the plot to read:

Line at $(x, \sin[x])$

You can see this is sort of an unraveling of the plot that looks like your original graph.



In fact, since your equation defines y to be sin x, you can change the plot to:

$$\text{Line at } (x, y)$$

...and the graph is the same. That is what Theorist creates when you make a new graph.

Plotting Multiple Lines on a 2-D Graph

Enter the following equations into a new notebook:

$$y = \sin(x)$$
$$y' = \sin(2x)$$
$$y'' = \sin(3x)$$
$$y''' = \sin(4x)$$

When you get a declarations dialog, make the y's user-defined variables.

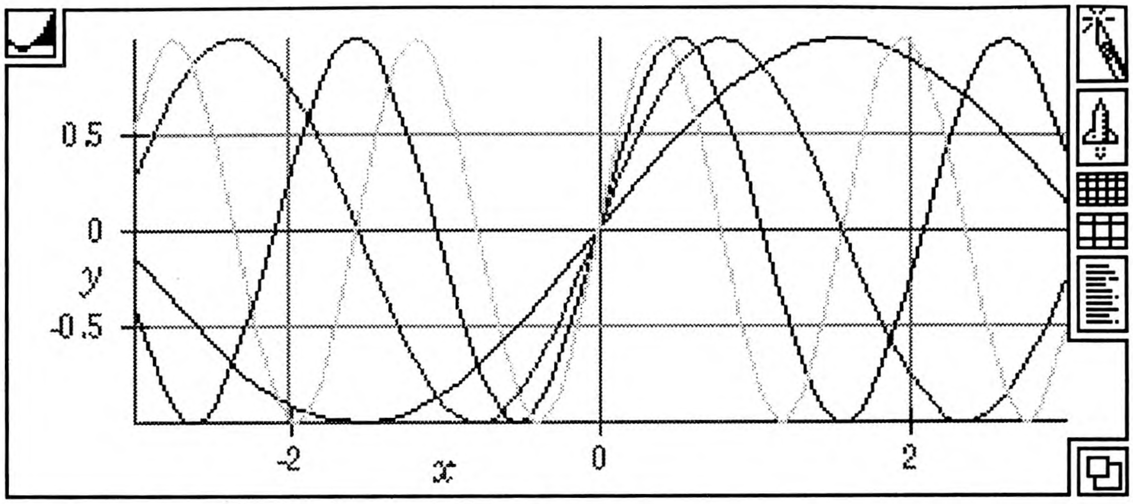
You want to look at all four of these at once, as a function of x. One way to do this is to make four graphs. Click anywhere in the first equation and click the palette button (or choose **Graph▸y = f(x)▸Linear**) to create a graph, then repeat for each of the remaining three equations.

Since graphs are just propositions, you can drag them around the screen and reposition them (as described previously in the Notebooks section). Rearrange the four graphs so they are stacked on top of each other like boxes.

This is one way of looking at all these functions, but it takes up lots of screen space, and you keep trying to visualize one graph overlayed on another. Making separate graphs as you now have is probably better for unrelated relationships, but in this case you want them all on the same graph.

Delete all four graph propositions. Click within the first equation and click (or choose **Graph▸y = f(x)▸Linear**) to make a graph of it.

Click within the second equation and choose **Add Line Plot** from the **Graph** menu's **Additional** submenu. (Accept the defaults in the axis variables dialog.) A new line is drawn showing the second function. Similarly, click within the third equation, add another line plot for it, and do it again for the fourth function. (Unfortunately, you can not use a multiple selection here, but you may be able to use the menu item's keyboard shortcut listed in Appendix A of the Reference Manual.)



Linestyles pop-up menu

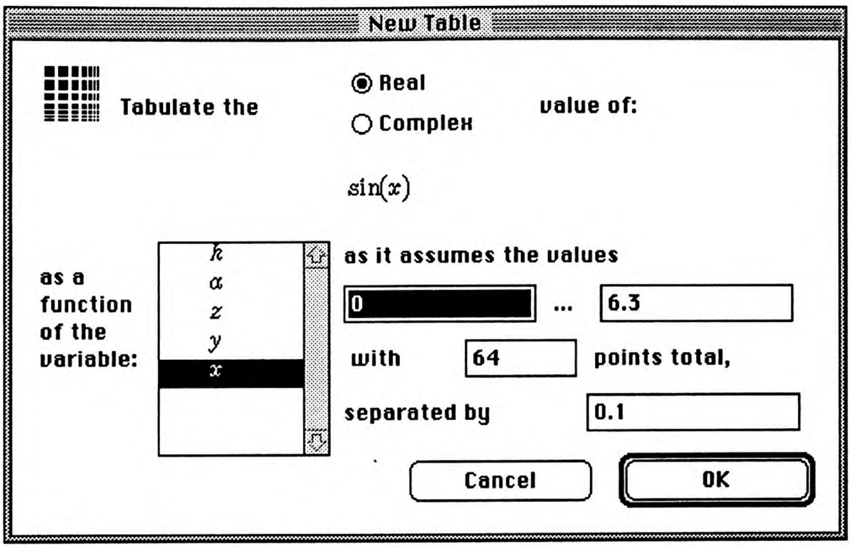
Theorist changes the color of each added line to distinguish between the plotted functions. (On black and white computers it changes the style of the lines.) Open the graph details and, with the pop-up menus, adjust the linestyles and/or colors for the four line plots.

Scatter Plots

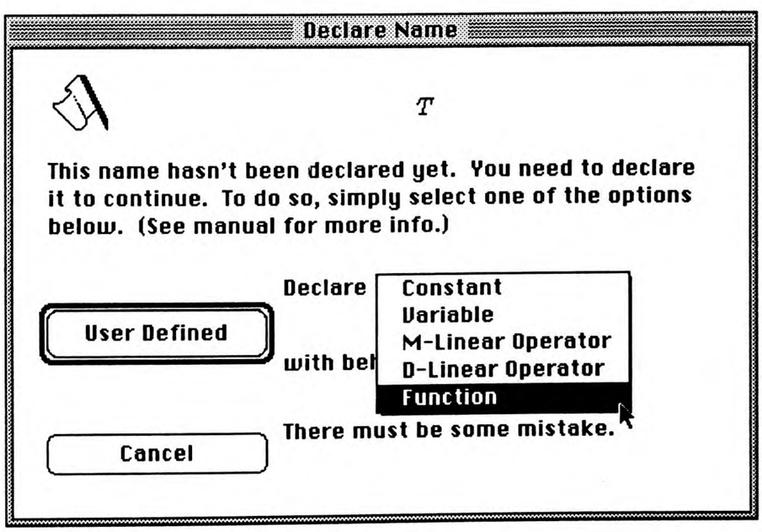
A scatter plot is a set of dots or markers that show individual data points. It is best for situations where you have to display several individual data values instead of a continuous curve or surface.

In a new notebook, enter sin(x) in the assumption .

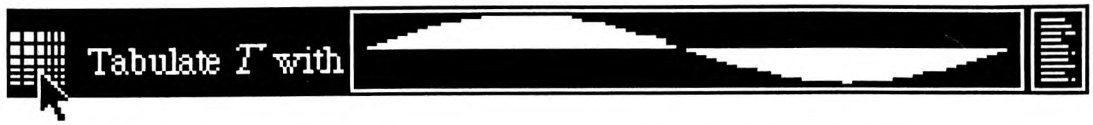
Select sin(x), then click (or choose **Manipulate▸Table▸Generate...**), which brings up the table creation dialog.



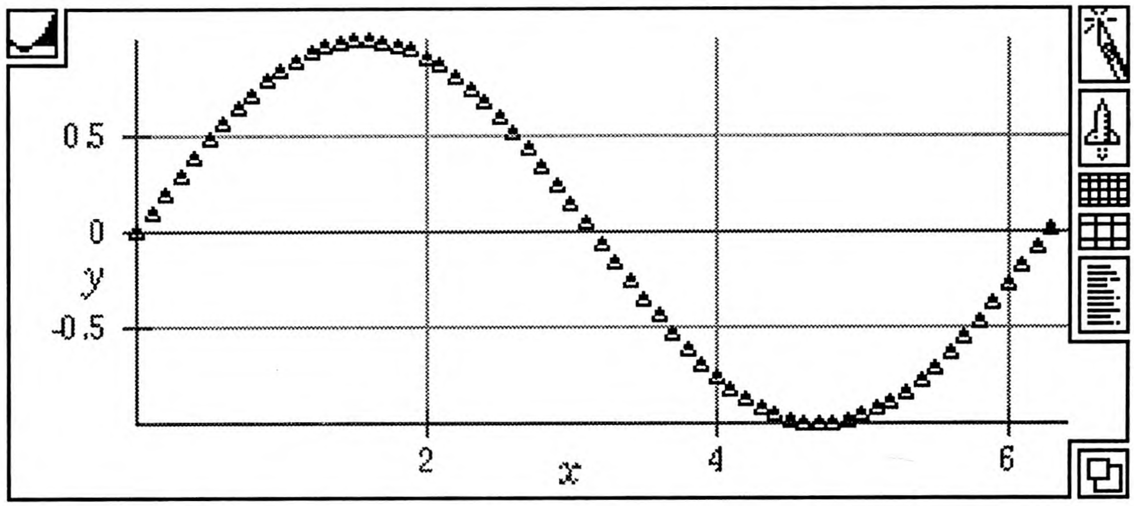
Accept the default settings. After the table is created, enter the name T for the table. Clarify the notebook and define T as a function.



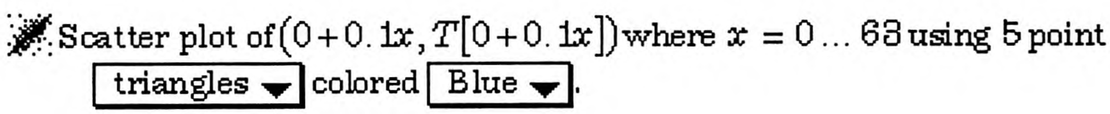
Select the table proposition by clicking its icon.



Click the palette button (or from the **Graph** menu's **Scatter** submenu, choose **Linear**). A dialog asks you to select the axis variables for this situation; use x and y and ignore the n parameter.



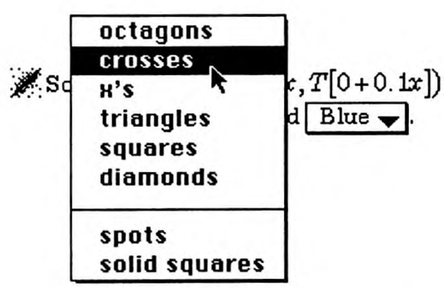
Open the graph details and examine the scatter plot proposition.



Plot Marker Styles

Change the style of the points in the scatter plot by choosing a different shape from the marker style pop-up menu in the scatter plot proposition.

The color of the points can be changed in a similar manner.



Select the "5" on the second line of the scatter plot proposition and type a number to replace it, which alters the size of the marker's diameter (in points). If you enter 0, the markers disappear. If you enter 1, the markers change to single dots, regardless of the marker style. If you enter any other expression, its numerical value is interpreted as the size. The expression can be a simple number or it can be an expression that depends upon the variable so that it changes size from one marker to the next.

There are 72 points per inch.

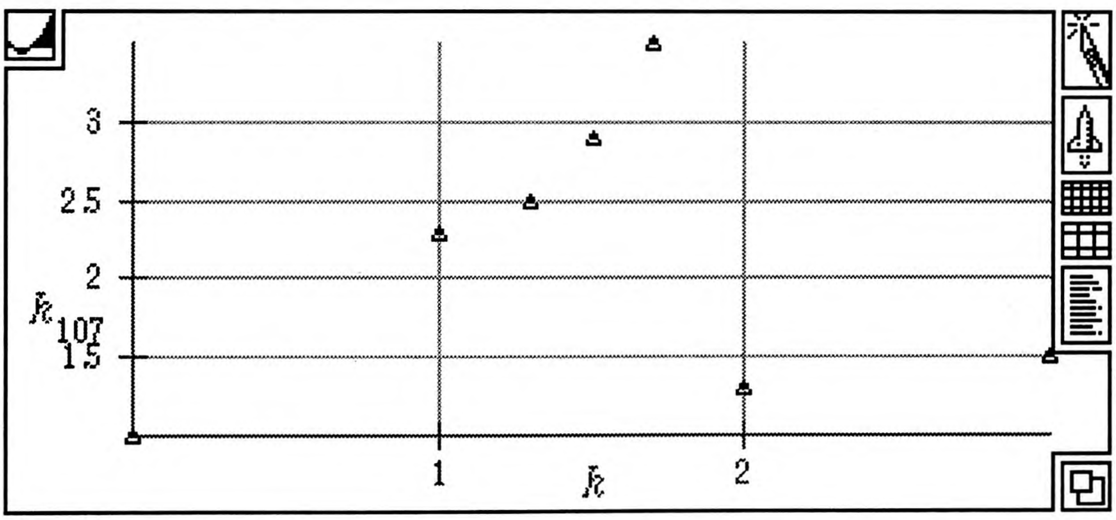
Scatter Plot from a Matrix

Use matrices to make a scatter plot any time the points do not come at regularly spaced intervals along the independent axis.

Create a new notebook and enter the following matrix.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 2.3 \\ 1.3 & 2.5 \\ 1.5 & 2.9 \\ 1.7 & 3.5 \\ 2 & 1.3 \\ 3 & 1.5 \end{pmatrix}$$

Clarify the notebook and declare A as an M-Linear Operator. Click anywhere in the matrix equation and click (or choose **Graph▸ Scatter▸Linear**). In the variables dialog, use k for both axis variables. In the declaration dialog, accept the predefined function declaration for RowsOf. The resulting graph follows.



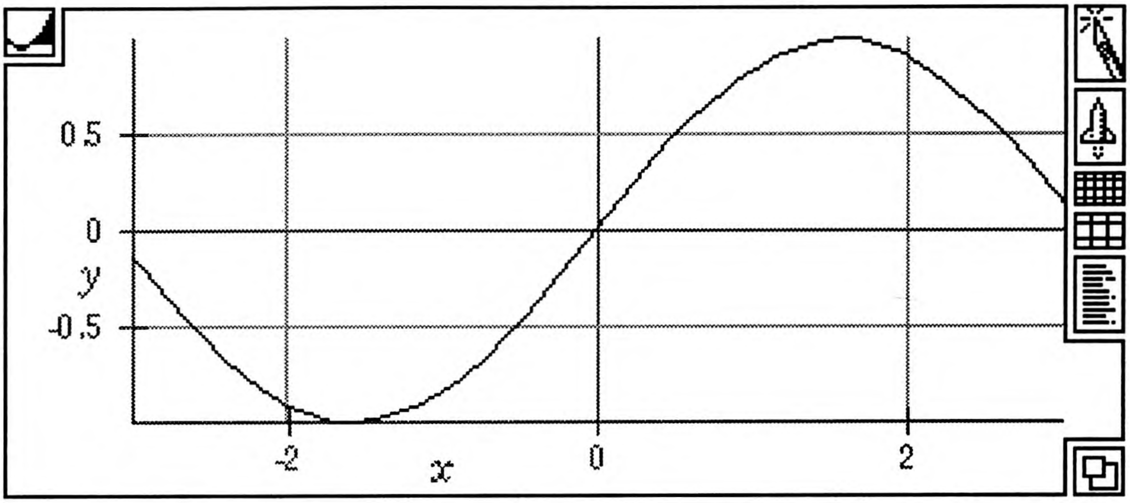
You can make other kinds of graphs by modifying the scatter plot proposition directly in the graph details. The scatter plot works similar to the line plot in the previous tutorial. The range variable (on the second line) assumes integer values from the range expression; for each integer value, a marker is drawn. The vector (top line) is the two-dimensional or three-dimensional location of the center of the marker.

DESIGNING FUNCTIONS

Distorting the Output

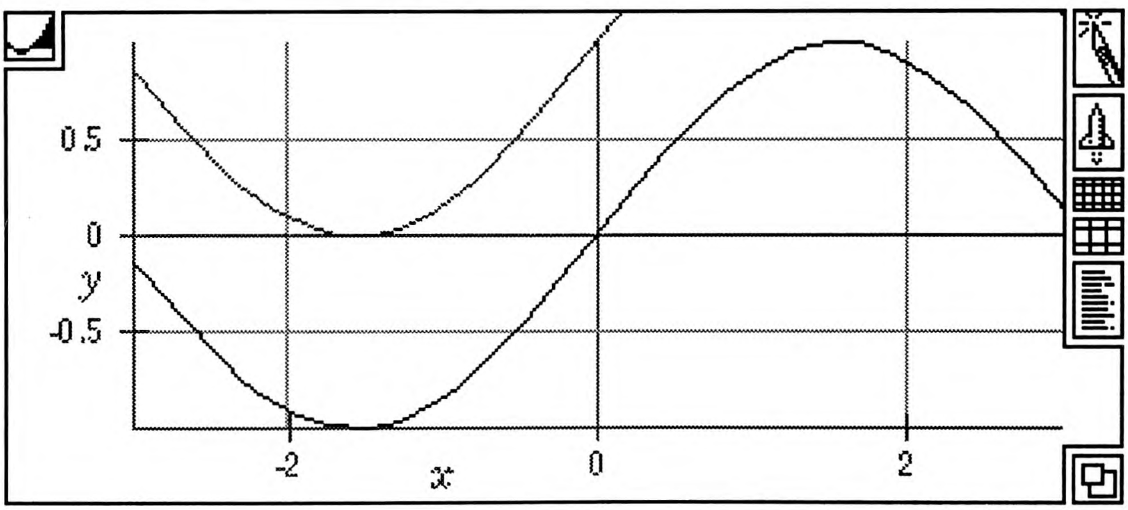
This tutorial describes several of the ways that you can modify the appearance of a plot by modifying the equation that describes the plot. The function used in this example is sin(x).

Start by typing **y=sin(x)**. Then click the palette button (or choose **Graph▸y = f(x)▸Linear**) to create a two-dimensional graph of the equation.



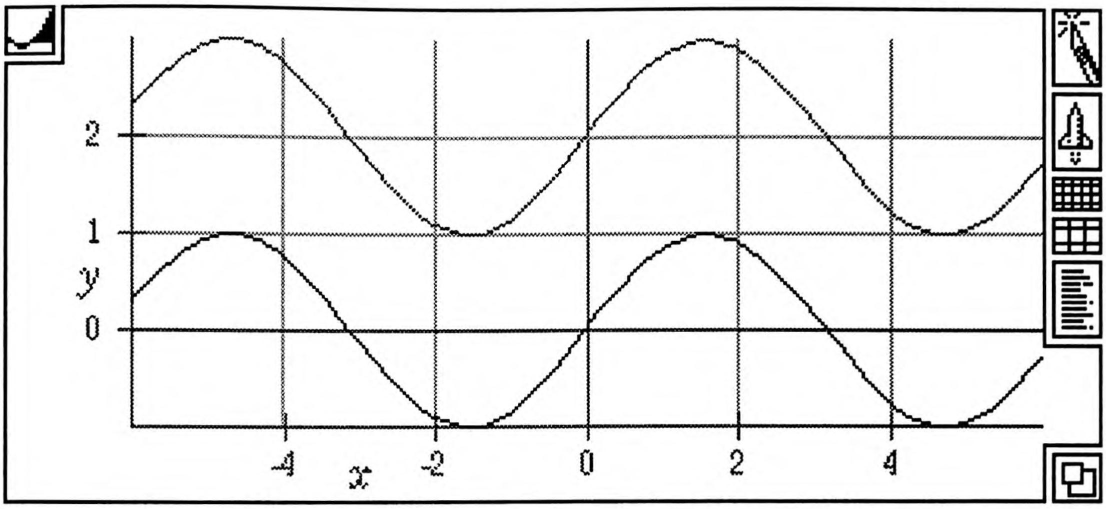
' is the single quote or apos-trophe.

To add a second plot to this graph, type [return] **y'=sin(x)+1**, then choose **Graph▸Additional▸Add Line Plot**. In the dialogs, declare y' as a variable, and use it as the y axis. Another, different-colored (or heavier) line appears in the 2-D graph.

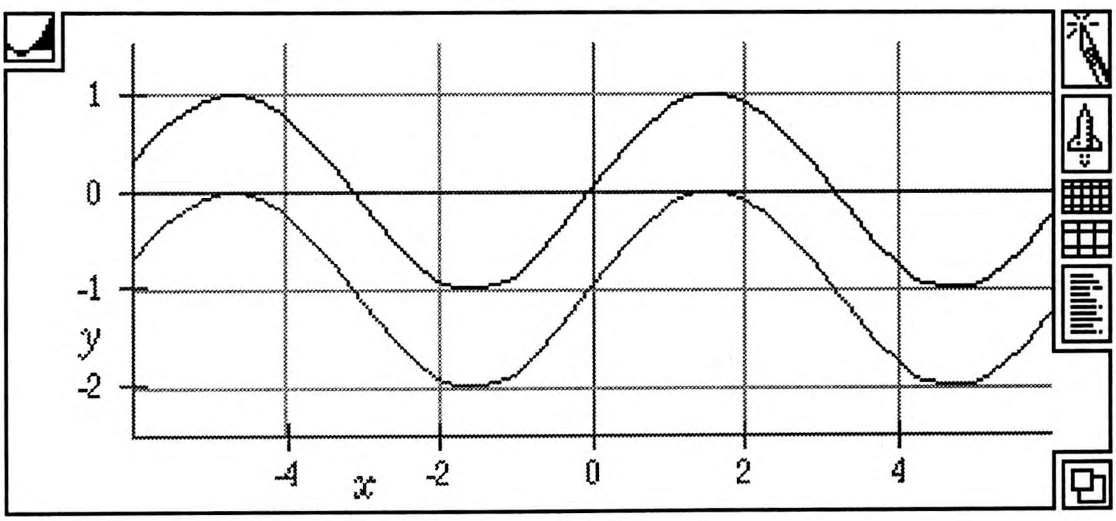


The black (or thinner) line is our original sin(x). The red (or heavier) line is the same thing, modified with the +1. Notice that the added red line is one unit above the original black line.

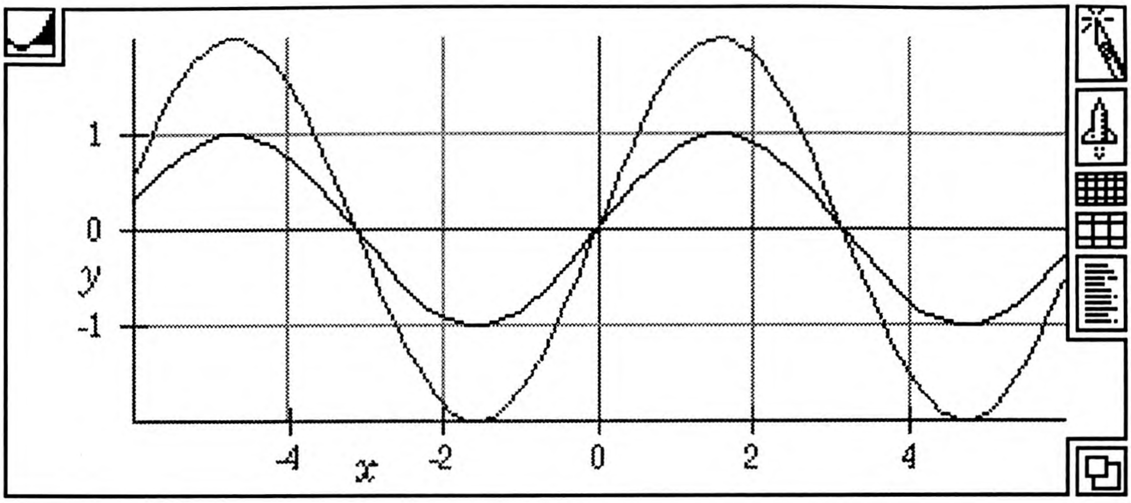
In the equation for y', change the 1 to 2 by selecting 1 and typing **2**. The graph redraws. You may have to zoom out with the Rocketship icon and you may want to change the view with the hand icon to get a better view.



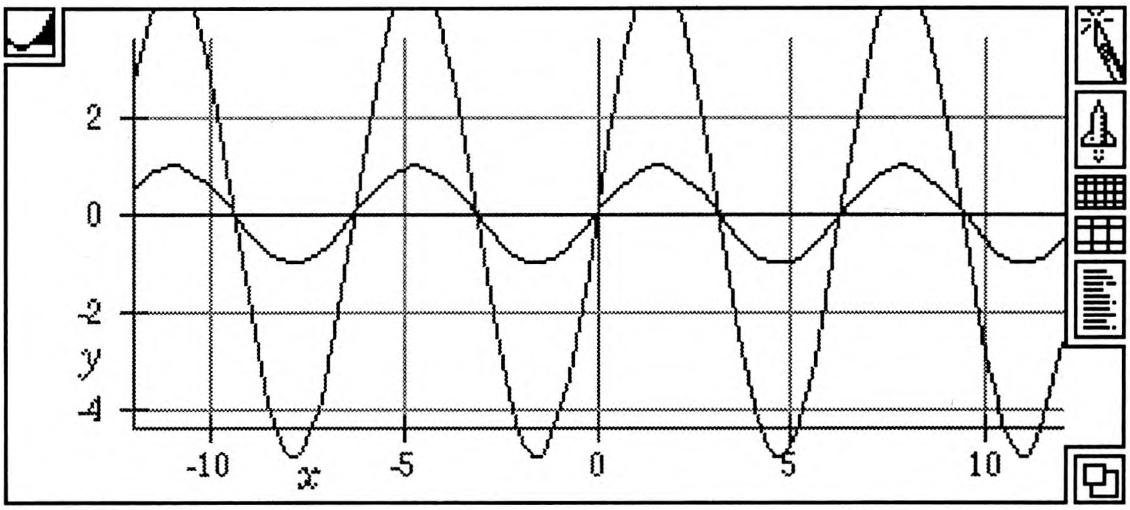
Now change 2 to –1 by selecting 2, pressing [delete], and typing **-1**. The graph lowers by one unit. (Again, you may need adjust the view.)



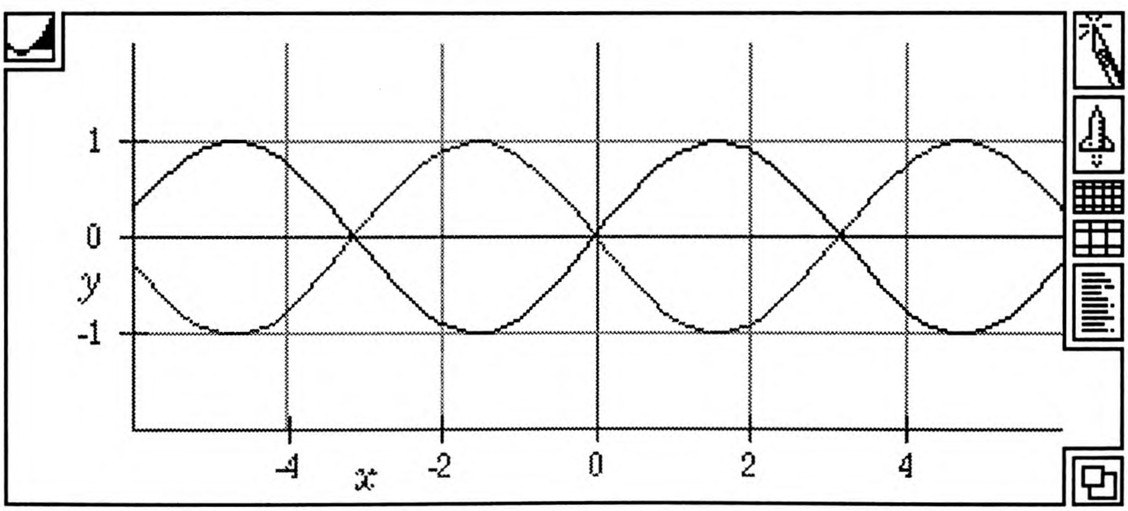
Now delete the –1. (Delete the extraneous "+?", too.) With sin(x) selected, type ***2**. The sine wave now swings twice as far because the *2 amplifies the sine wave by a factor of 2. (Again, you may need adjust the view.)



Change 2 to 5 and the effect intensifies even more. (Zoom out for a better view.)

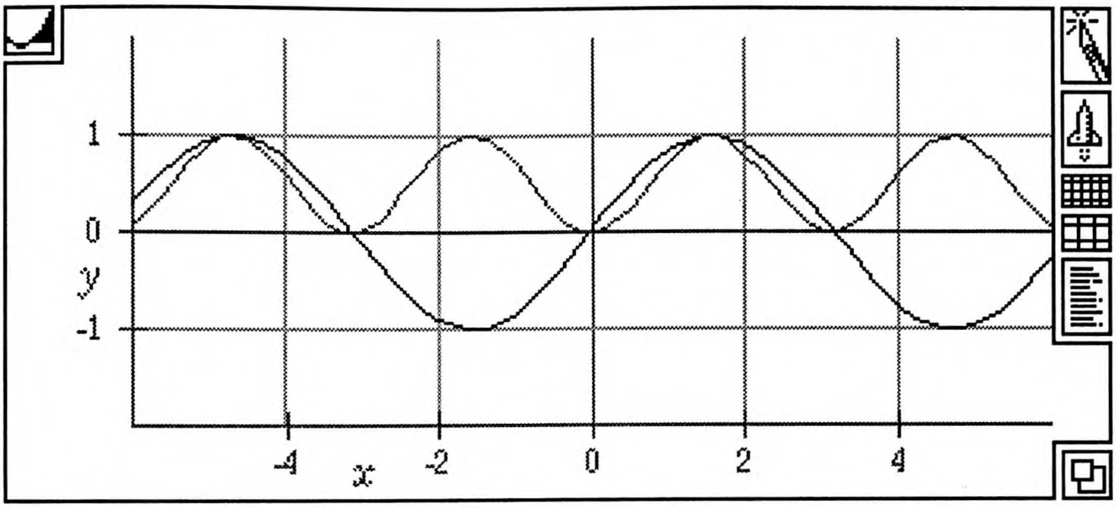


Get rid of the *5 and change sin(x) to –sin(x). (Select sin(x) and press the minus key to negate it.) This turns the graph upside-down. Hold option (Macintosh) or alt and Ctrl (Windows) and click the Rocketship icon to zoom back.



All of these modifications so far are linear; you could accomplish the same thing by drawing the graph on a rubberized sheet of paper and stretching it vertically or turning it upside-down. Now let's try some non-linear transformations.

Delete the $-\sin(x)$ and type **(sin(x))^2**. (You may have to Clarify or click the graph to make it redraw.)



Here are the effects of squaring:

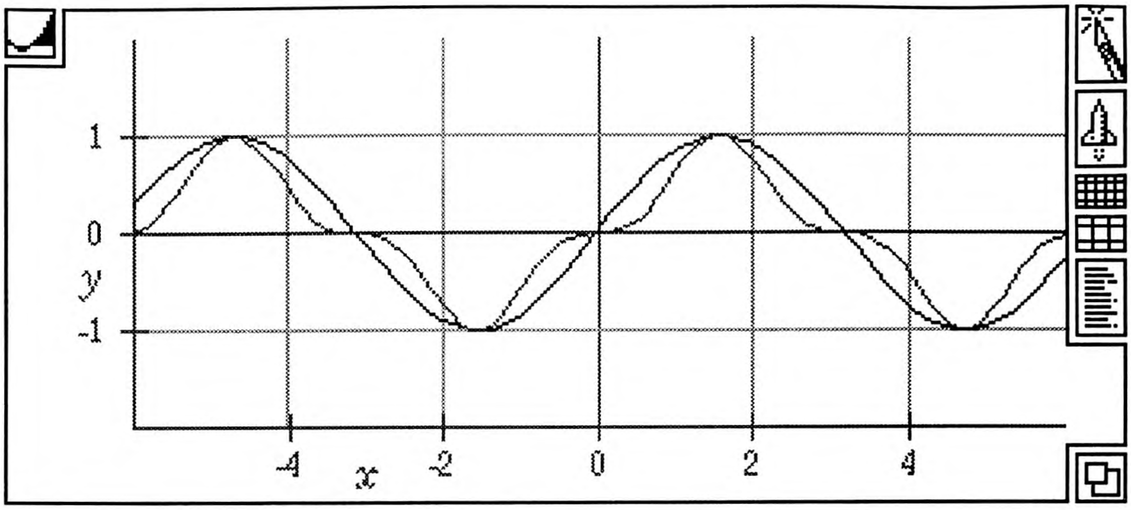
- the square is always positive
- if the value is less than 1, the square is even smaller
- if the value is greater than 1, the square is even larger (not observed here)

The result is that all the negative values are now positive and all of the places that zoomed through zero now form rounded curves as they bounce up again. The whole thing looks just like another sine wave. In fact, it is! There is a well-known trigonometric identity which describes it.

Exercise

By looking at the graph of $(\sin[x])^2$, deduce the trig identity without using any algebra and without cheating and looking it up.

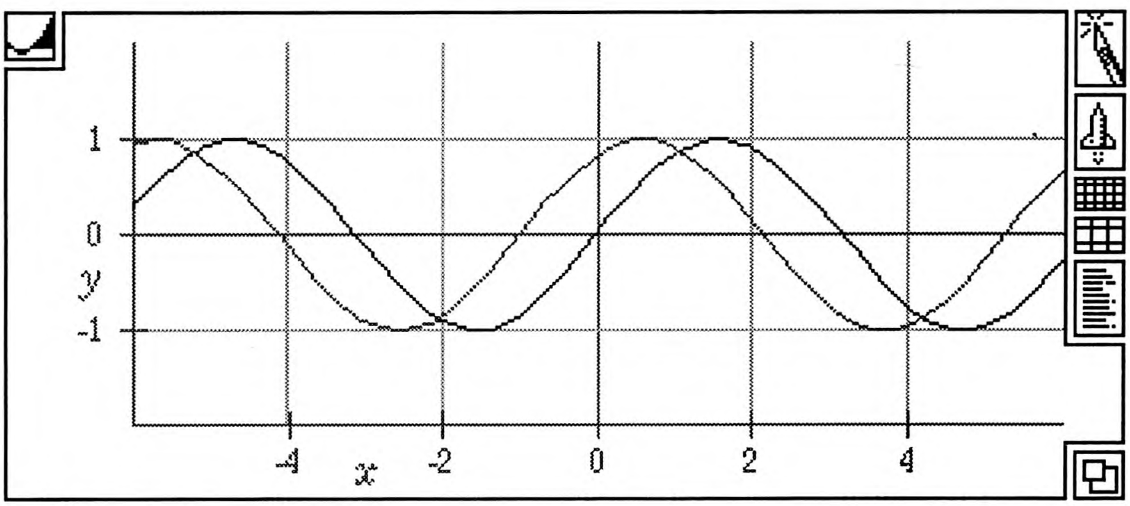
Now change $(\sin[x])^2$ to $(\sin[x])^3$. This is a very nonlinear transformation!



Distorting the Input

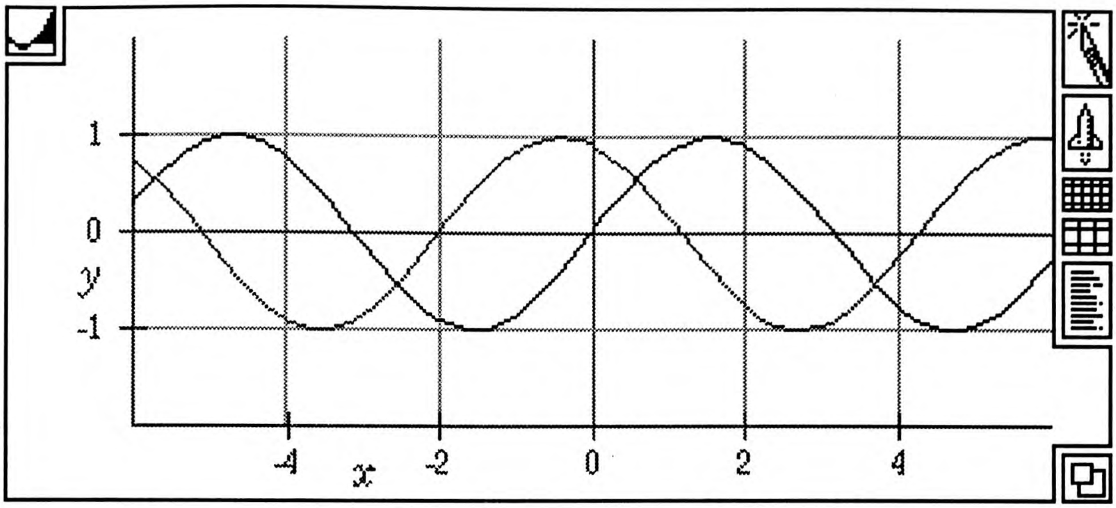
That's sin(x + 1), not sin(x) + 1.

We just saw what happens when you distort the result of a function. Now we will see what happens when we distort the input. Not only will the effects be horizontal rather than vertical, but the effects will also be the reverse of what you would expect if the same distortion were applied to the output. Change $(\sin[x])^3$ to $\sin(x + 1)$.

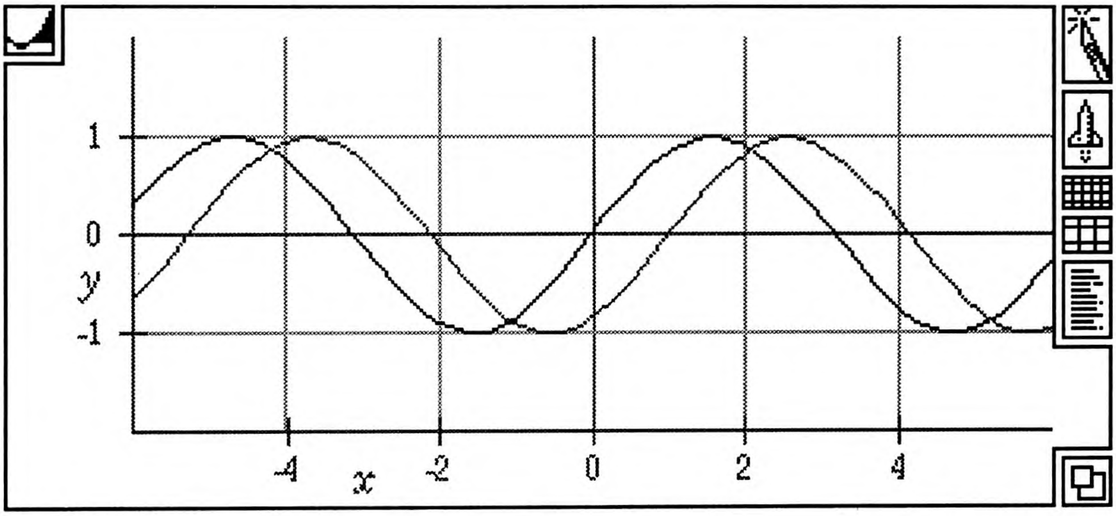


You can see that the +1 has the effect of moving the function over by one. But notice it is in the opposite direction from what you might think—it was moved one unit *backwards*. Think of it this way: in order to get sin(0), where the function crosses zero, you have to make x be –1 to cancel out the +1.

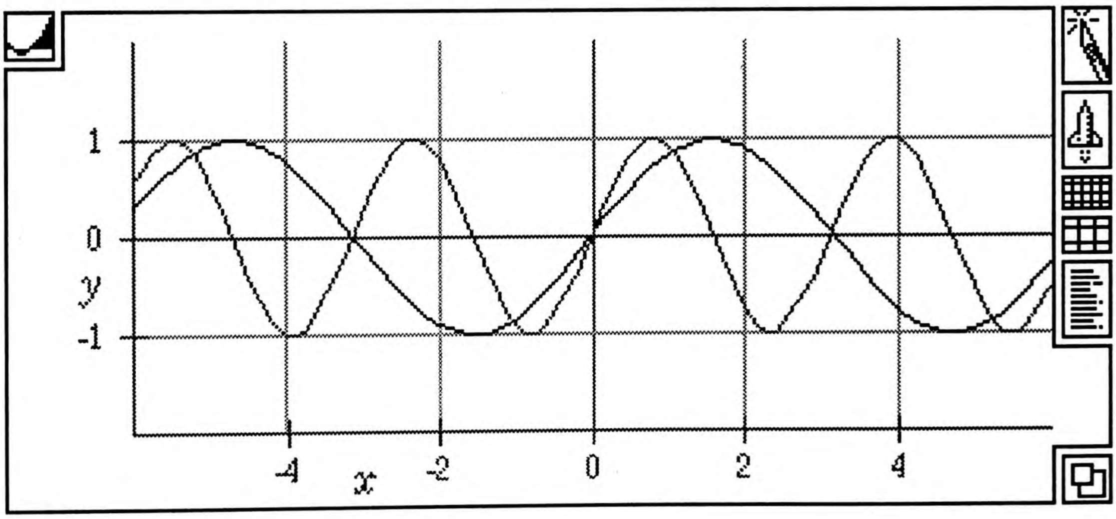
Now select the 1 and change it to 2. This moves it over the way you expect.



Now change the 2 to –1 by selecting the 2, pressing [delete], and typing **-1**. Now the curve moves forward. Remember that in order to move it from zero to some place *n*, you must subtract *n* from *x*.

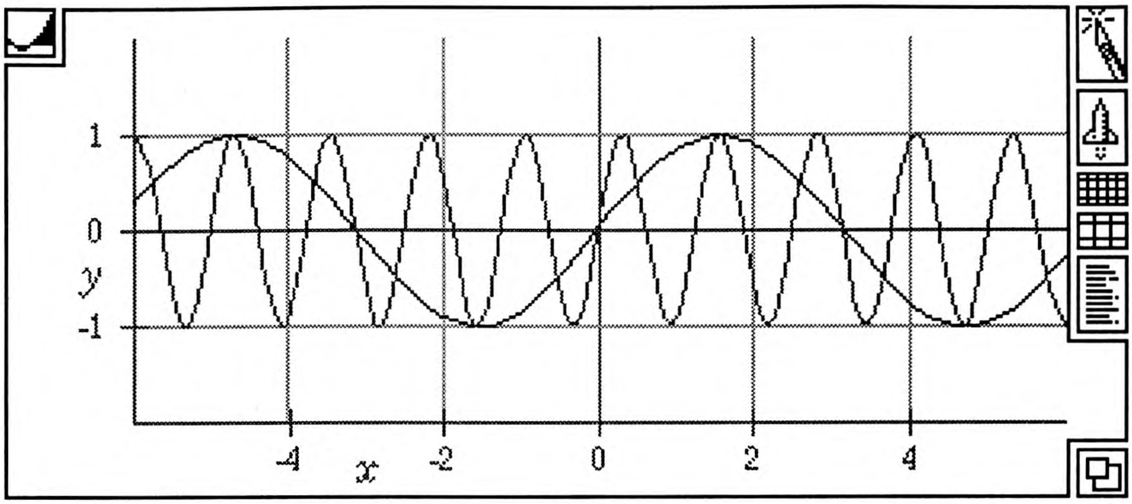


Delete the –1 and any extraneous ?, then, with *x* selected, type ***2**.

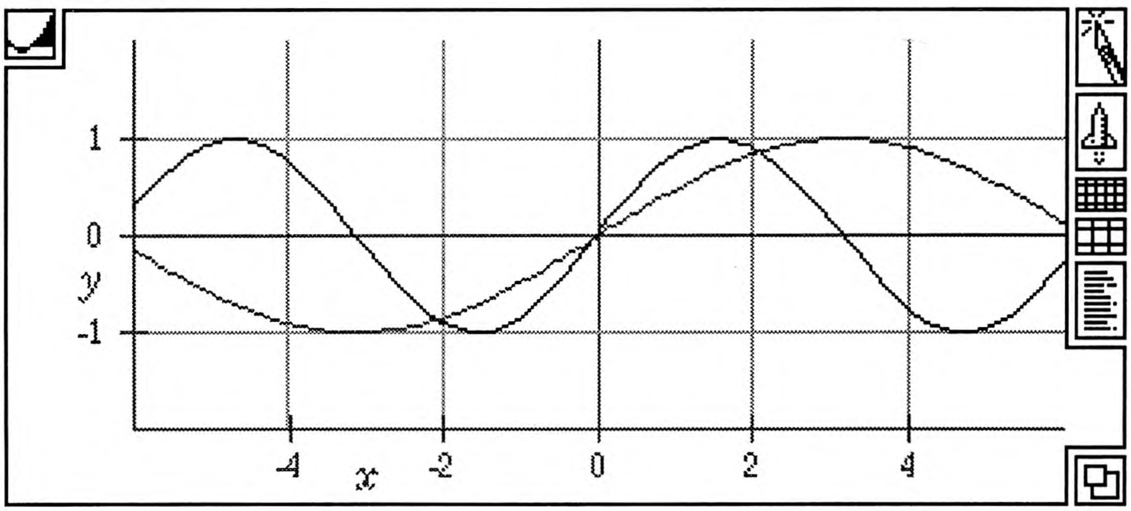


Notice that the higher frequency graph has more humps. Perhaps you expected the graph to expand out by a factor of two instead of compressing. Well, sin() is the same—you are just making x change twice as fast.

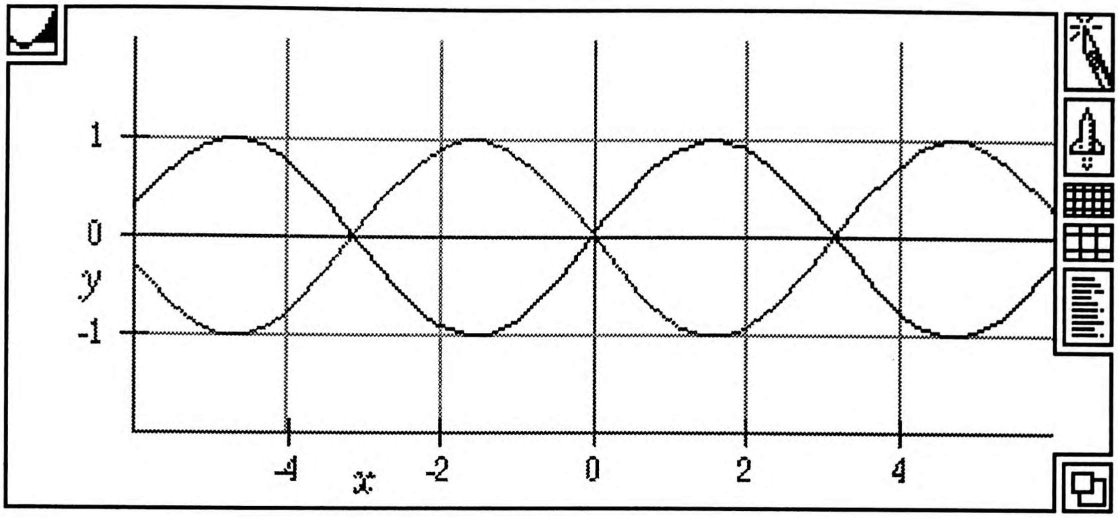
Now change the 2 to 5. Note that the effect intensifies:



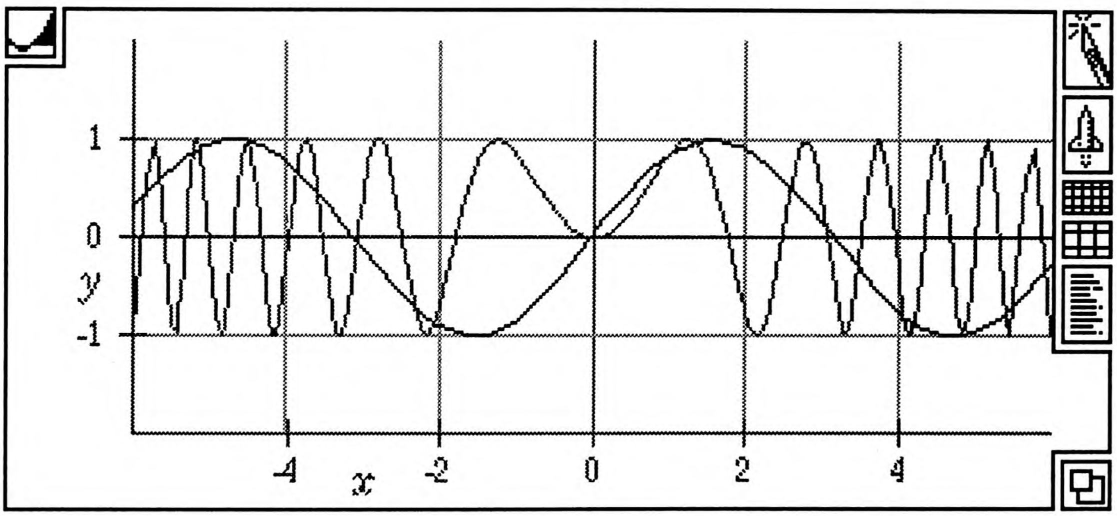
Change the 5 to $\frac{1}{2}$ (select 5 and type **1/2**) and notice what it looks like:



Now delete the fraction and any extraneous ?, then, with x selected, type **-** to negate it, so the expression appears as sin($-x$). Note the function is turned upside down.



Select $-x$ and type **-**. (Negating a negative makes it positive.) Type **^2** to make x^2.



You can see the effect of x^2 in the behavior near $x = 0$. It starts off very slowly, in the positive direction, whether x is positive or negative. After a while, though, x accelerates and you get a wildly varying function. It is almost as if you have sin($5x$) except the 5 is not 5, it is x, getting bigger and bigger as x gets larger.

COORDINATE MAPPING

This tutorial describes how to use one of the predefined functions, FromPolar, to alter the coordinate system—and thus the appearance—of a parametric plot.

When you map a graph to a new coordinate system, you end up with a graph that bears a resemblance to the original graph, but is distorted. That's the whole idea behind the new coordinate system.

A linear coordinate system can take the original image and move it, rotate it, and do certain kinds of uniform stretching on it. You can do a linear transformation by simply multiplying and adding matrices.

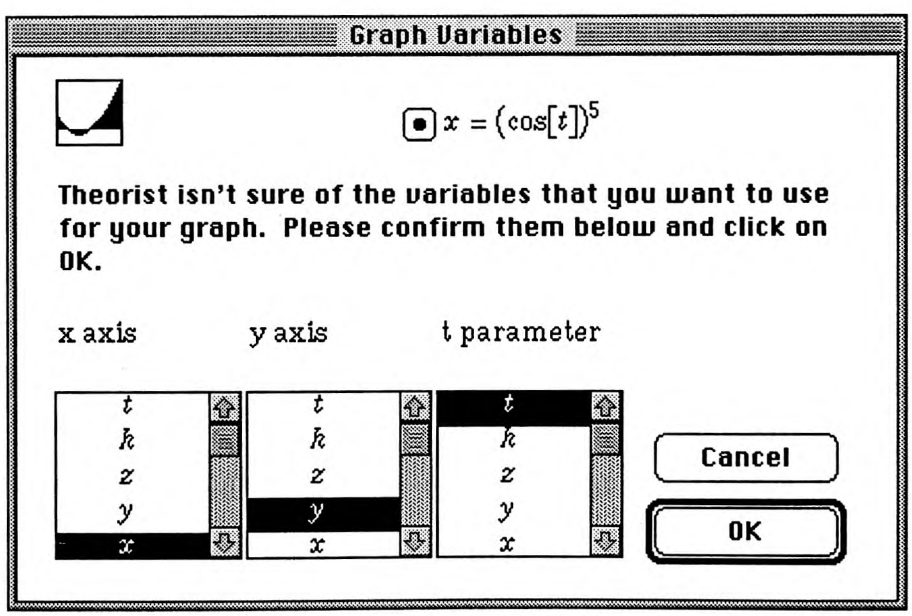
In this tutorial, you use a nonlinear coordinate system known as Polar Coordinates. The resulting distortion is interesting.

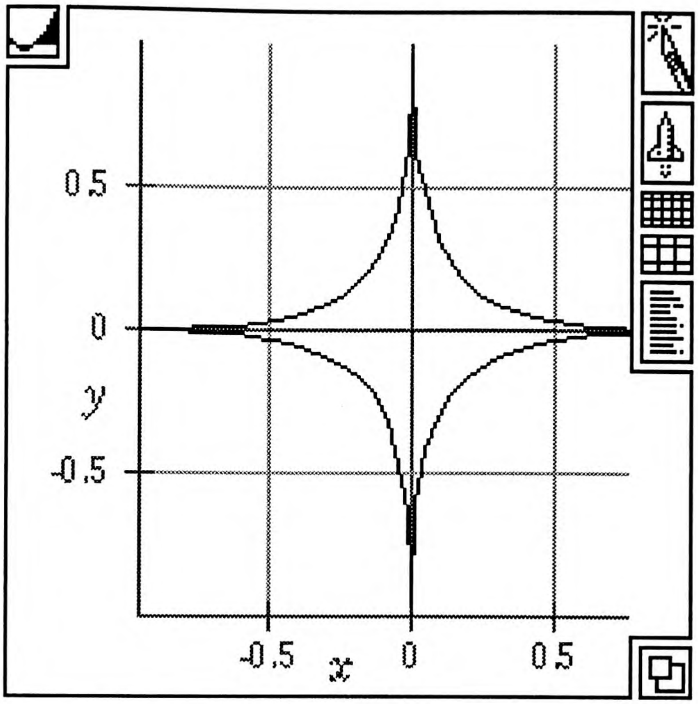
First, let's make a 2-D graph of a 4-point star. Enter the equations:

$$x = (\cos[t])^5$$

$$y = (\sin[t])^5$$

Choose **Clarify** from the **Notebook** menu and declare *t* as a variable. Select both equations and choose **x = f(t), y = g(t) Parametric** from the **Graph** menu's **Other** submenu to make a parametric graph. When Theorist puts up a dialog asking about axis variables, match *x, y,* and *t* to their axes.






Graph Icon

Let's make another graph with this same star mapped through polar coordinates. Click on the graph proposition icon. Perform a copy, then paste. This duplicates the graph and places the copy below the original.

Now, open the bottom graph's details and find the line plot:

Line at (x,y) where $t = -3 \ldots 3$ with a normal▾ line, colored Black▾.

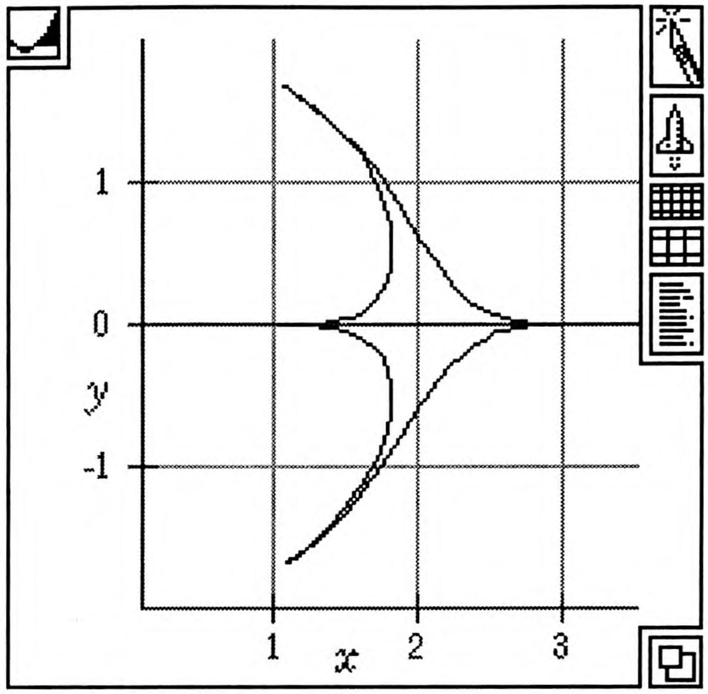
Our next step is to adjust (x,y). Click down so the blinking cursor appears just before the first parenthesis in (x,y). Type **FromPolar**, exactly with initial caps and no space between From and Polar:

Line at FromPolar(x,y) where $t = -3 \ldots 3$ with a normal▾ line, colored Black▾.

Clarify and when the dialog appears, click **PreDefined** to declare FromPolar as a predefined function. Now the graph looks like a mess, but don't worry about it. It's a mess because the center of the star is right on top of the origin. The origin, in polar coordinates, is a very strange place where distortion hits an all-time maximum. Let's move the center away from the origin...

The star, which was in the (x,y) rectangular coordinates, is now in the (r, θ) coordinate system. (You still use the variables x and y but the way you are looking at it, the x is interpreted as r and the y as θ). You can improve the appearance of your graph by moving it 'up' in the r direction. That is, by adding something to r (the first coordinate).

Change FromPolar(x,y) to FromPolar(x + 2, y). You will have to adjust your view of the graph with the hand and the Rocketship because the graph is off to the right.



Exercise

Change the 2 to 3 or 4 to move the star's center further away from the origin. Slide it around by adding something to the θ coordinate (actually y). This rotates it around the origin.

IMPLICIT EQUATION GRAPHS

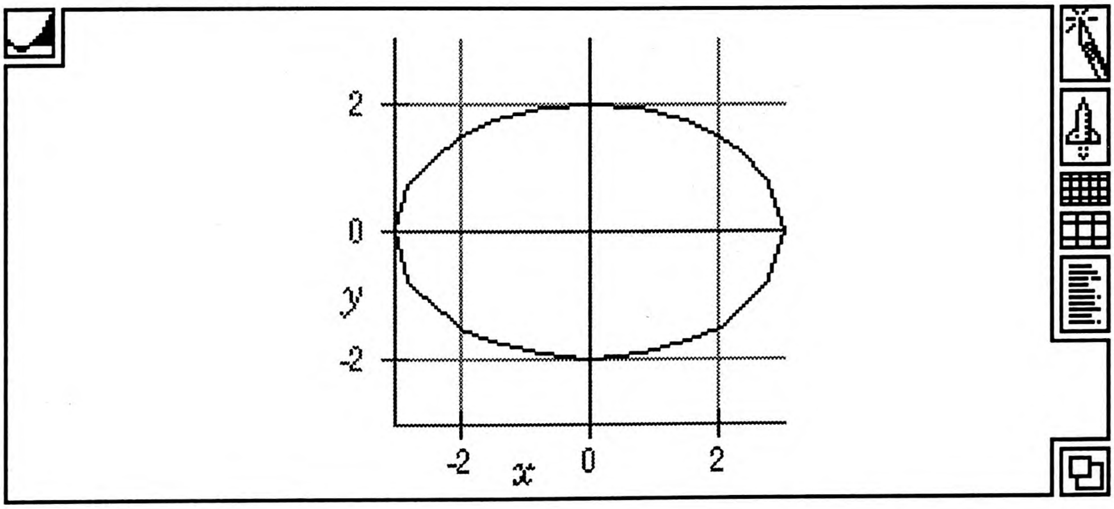
Sometimes, equations in two variables cannot be resolved into the standard explicit form, like y = [some expression in x]. These are called implicit equations. Converting to an explicit form can be a tedious—even impossible—task. Fortunately, Theorist can plot implicit equations directly.

Create a new notebook and enter the equation for an ellipse.

$$\square \; \frac{x^2}{9} + \frac{y^2}{4} = 1$$

Note that this equation can not be solved explicitly for y, because it would either be multi-valued or non-valued. (If x is 4, then y is imaginary and would not show up on the graph. If x is 0, then y can be either +2 or −2; there is no way to make an expression yielding two answers like that.) Graphing the resulting function would require splitting into two parts (an upper and lower half). This can become tedious depending upon the complexity of the function. Using the implicit function feature simplifies plotting functional relationships of this type.

With a selection in the equation, pull down the **Graph** menu and from its **Other** submenu, choose **f(x,y) = g(x,y) Implicit**. Accept the default axis variables in the dialog. Open the details and change the aspect ratio from Stretch to Fit to True Proportions.



Notice the graph is not as smooth as with a typical graph, such as a graph of sin(x). This is because implicit equations are much more difficult to calculate.

With explicit equations, Theorist chooses different values for x and calculates the value of y to draw a line. The result is an (x, y) pair guaranteed to be on the line exactly. It is a one-step operation where forty to 100 evaluations generally get as much smoothness as is necessary.

With implicit equations, Theorist chooses *pairs* of values (x, y). It evaluates both sides of the equation for each data point (x, y) to see how close to zero the result is. Rarely scoring a direct hit, Theorist then does linear interpolation to estimate where the line lies. The resulting line segments are merely rough sketches of where the equation is true. By default, eighty-one (x, y) pairs are evaluated, yielding the equivalent of nine data points vertically and nine horizontally. To get a resolution approaching that of typical line plots at normal resolution, you must increase the implicit graph resolution to 64 or more, which is time-consuming to calculate.

Important Note

Implicit graphs do not guess the domain and range as well as some of the other kinds of graphs. This may require you to manipulate the graph using the Rocketship or the Knife (or editing the values in the graph details).

PLOTTING WITH LOGARITHMIC SCALES

Create a new notebook and enter the following equation.

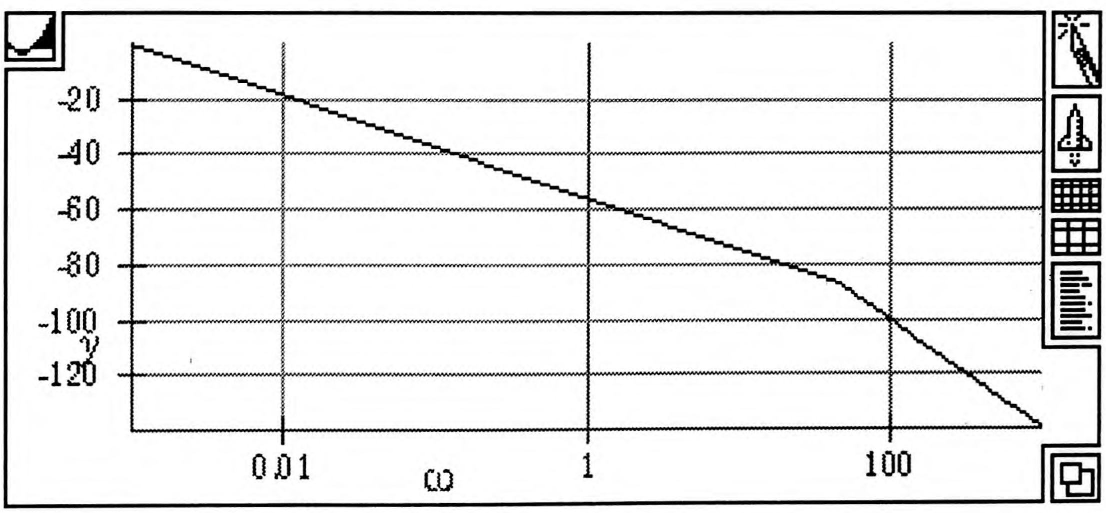
$$\square\ y = 20\log\left(\left|\frac{1}{\left[1+2\zeta\left(i\frac{\omega}{\omega_n}\right)+i\frac{\omega}{\omega_n}\right]^2}\right|\right)$$

This equation is used to see the second order linear frequency response of the system.

Clarify and declare the symbols ζ and ω as variables. Create equations to give values for ζ as 1 and ω_n as 1.

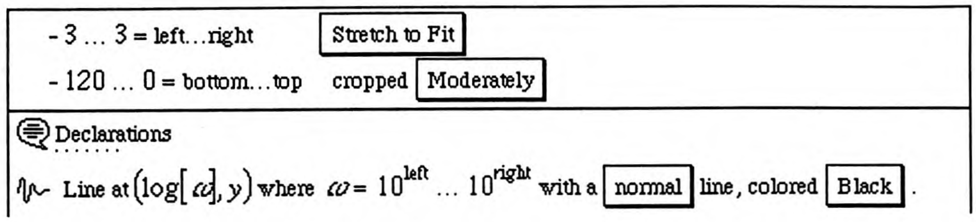
$$\square\ \zeta = 1$$
$$\square\ \omega_n = 1$$

Click in the original equation. Create a logarithmically scaled graph of y by choosing **Graph▸y = f(x)▸SemiLog-X**. In the familiar dialog, use ω and y as the axis variables.



Suppose we are only interested in ω ranging from 0.1 to 10. We have two ways to change the domain of ω: use the knife or edit graph details. In this example, we will edit the graph details to develop a better understanding of how logarithmic axes are used in Theorist.

Open the graph details.



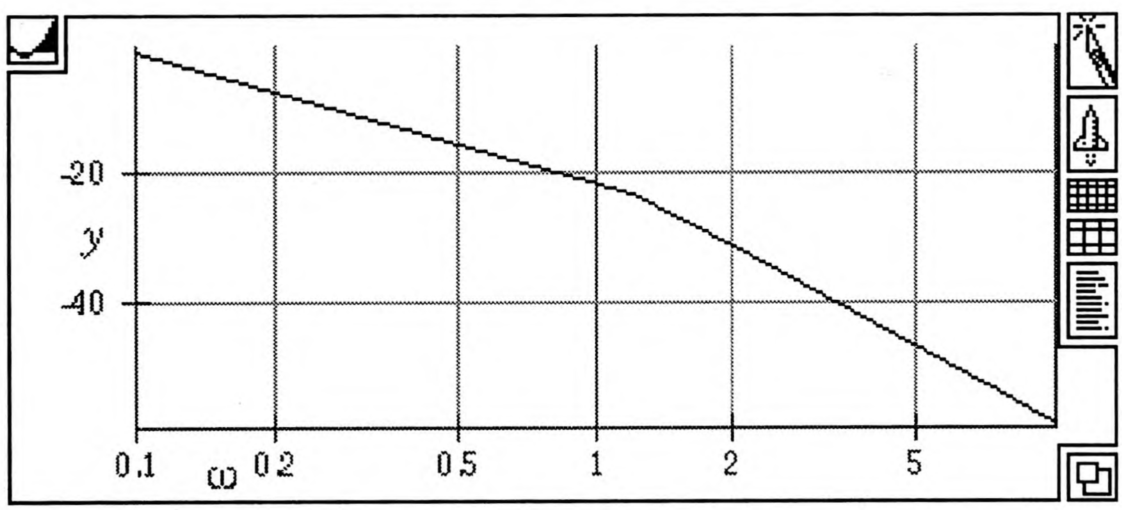
Note how left and right dictate the region plotted in the graph, but in linear coordinates which are logarithms of the frequencies, not the values of the frequencies themselves. To change the domain of ω to 0.1 to 10 (which is 10^{-1} to 10^{1}), edit the statement for left...right so it becomes –1 for left and 1 for right. While you are in the details, go ahead and change the bottom value to –60 to fill up the graph.

The vertical scale is linear and so the numbers represent actual values.

$-1 \ldots 1 =$ left...right

$-60 \ldots 0 =$ bottom...top

After making these changes, the final graph is drawn as shown below.



Theorist also supports SemiLog (y axis is logarithmically scaled) and Log-Log (both axes scaled logarithmically) plot styles.

THREE-DIMENSIONAL GRAPHING

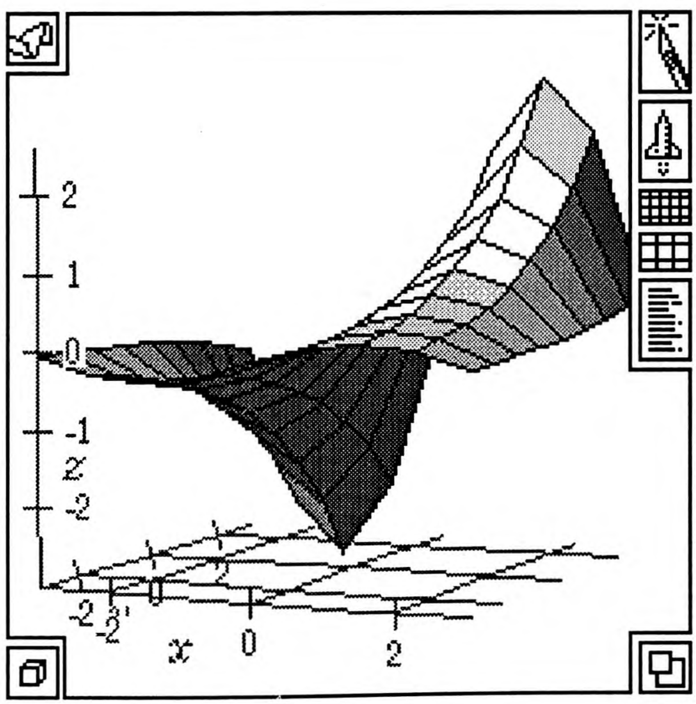
This tutorial describes a few basic tools you can use to interact with three-dimensional graphs, including:

- The hand cursor, for rotating the graph directly
- The Rocket, for zooming out and in on a graph
- The Knife, for selecting a portion of a graph
- Resolution accuracy buttons for smoothing a surface or speeding drawing

You were introduced to these tools in the chapter on two-dimensional graphing and now you will see how they are applied to 3-D graphs.

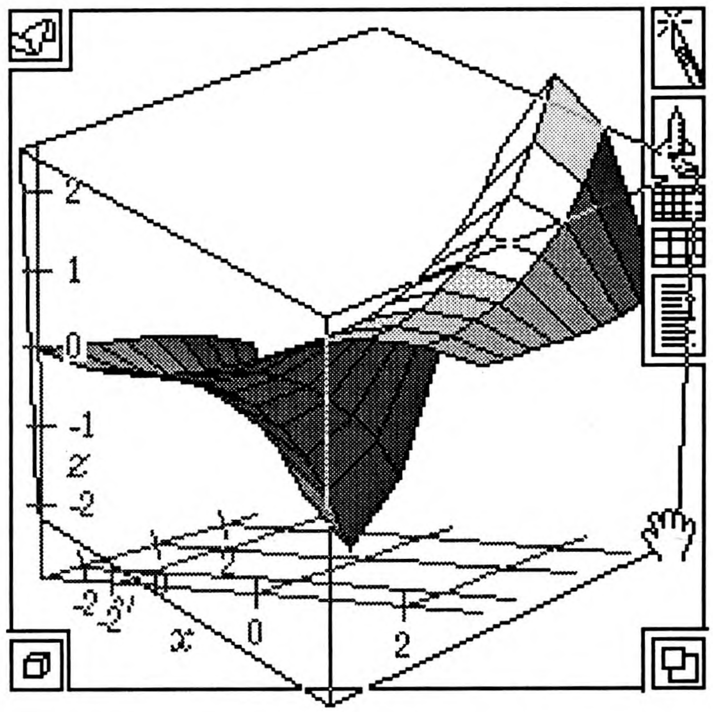
Type this formula into a new notebook: **z=1.5^y** [esc] ***sin(x)**.

With the blinking cursor in the equation, pull down the **Graph** menu and from its **z = f(x, y)** submenu, choose **Illum. 3D**. A dialog appears asking you to confirm the axis variables you want for your graph. Match the variables to their axes and click **OK**. Then a three-dimensional graph theory appears with a surface plot.

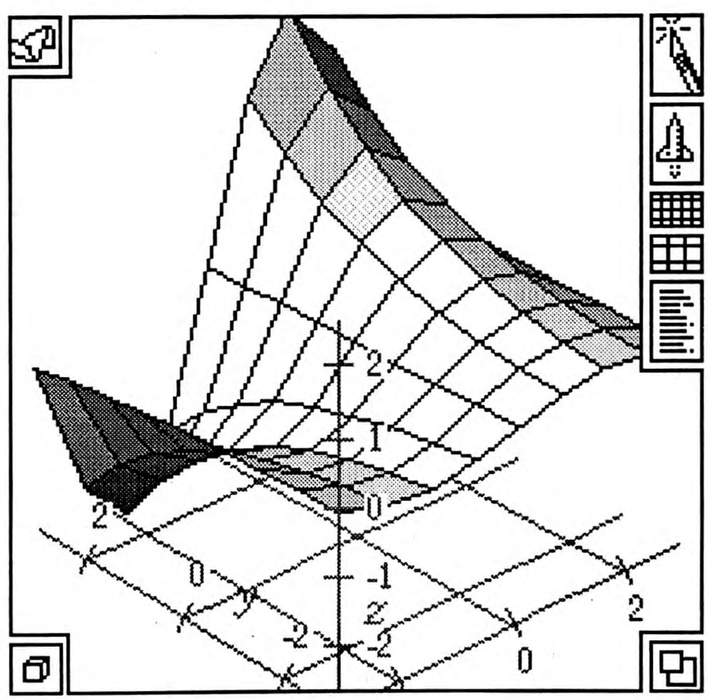


Whether Theorist continually redraws the graph or just draws an outline depends on the current resolution and the resolution setting in the graph preferences dialog.

As with two-dimensional graphs, the cursor becomes a hand for scrolling the graph. This time, though, what it does is a bit different. As you click and drag the image left, right, up, or down, the portion of three-dimensional space displayed in the viewport rotates as if you had your hand on a globe. As you drag, the graph either redraws with the new orientation, or a box outline shows the bounds of the graph as you drag,



...and the graph redraws with the new orientation when you release the mouse.



The outline box line width increases at each multiple of 15 degrees of rotation (in latitude and longitude). If you rotate the cube by multiples of 15 degrees longitude *and* 15 degrees latitude, the line becomes even heavier.

Rotational freedom is limited to rotating the display around its vertical axis, and rotating the vertical axis towards and away from you. Hold the [option] key (Macintosh) or [alt] and [Ctrl] keys (Windows) as you drag to tilt the vertical axis and rotate the graph more freely. See the Theorist Reference Manual for more information.

Momentum is not animation.

Note: rotational momentum occurs in the background when you have left Theorist to use other programs. You may want to stop a rotation by clicking on the graph before putting Theorist in the background.

If you click to grab a graph, move the hand, and release the mouse button while the hand is still, the graph redraws in the new orientation. However, if you click and grab a graph, move the hand, and release the mouse button before you have stopped moving the mouse, the graph is continually redrawn along the desired rotation path. The grab-and-stop method is like someone slowly moving a globe to position it at a specific location, whereas the grab-and-swipe method is like a five-year-old kid giving the globe a good swipe to watch it spin. The amount of momentum is determined by how short or long the "swipe" is—grab a corner and swipe to the opposite corner for lots of momentum or grab in the middle with a short swipe for a less radical rotation. (The former method may be more entertaining, the latter more enlightening.)

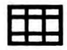
Orientation Guide (showing top view)

A feature unique to 3-D graphs is the orientation guide in the lower left of the graph proposition. It displays a cube which you can think of as your 3-D graph space. The cube by itself indicates you are looking at your graph's original orientation. The cube with an arrow pointing to one of its sides indicates which side you are looking at (in case you get dizzy or otherwise disoriented while rotating a graph). Click on the icon to return the graph to its original orientation, saving yourself the trouble of rotating it back in place.

More Accuracy Button

Notice how the graph is made up of flat panels, which is disappointing because in reality it is a smooth surface. The computer must evaluate at points to get a good guess. We can improve the guess. The More Accuracy button increases the number of data points for a higher resolution plot, which makes certain types of plots smoother. As you click the button, notice that two numbers appear next to it. The left number is the current resolution, and the right number will be the new resolution when the graph is redrawn. Right now, increase the resolution a few clicks.

Rotate the graph to see the effects of the higher resolution. The more detailed graph looks pretty, but it slows down the drawing speed (so much that Theorist will not continually redraw the graph and just draws a box outline while rotating). Higher resolution makes nice looking pictures for publication but you will probably want to keep the accuracy low while working with Theorist, no matter how fast your computer is.



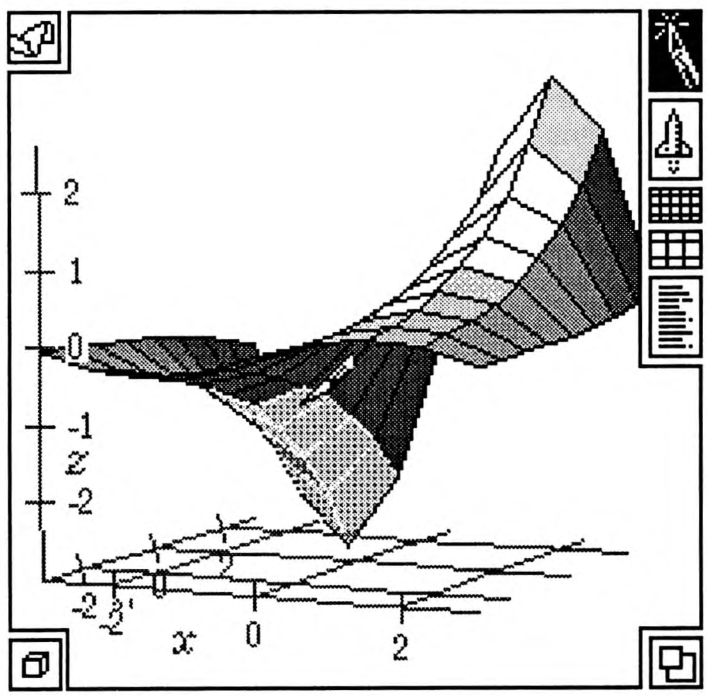
Less Accuracy Button

The Less Accuracy button has the opposite effect of the More Accuracy button. It decreases the number of data points for a lower resolution plot, which has the unfortunate side effect of making certain types of graphs more chunky or jagged. This is especially true with this particular graph. Give the Less Accuracy button several clicks now and take the resolution all the way down to 1. Notice the less appealing plots as you go. Click the More Accuracy button a few times to bring the resolution back to its original setting of 8. Notice the slower drawing speed as you go.

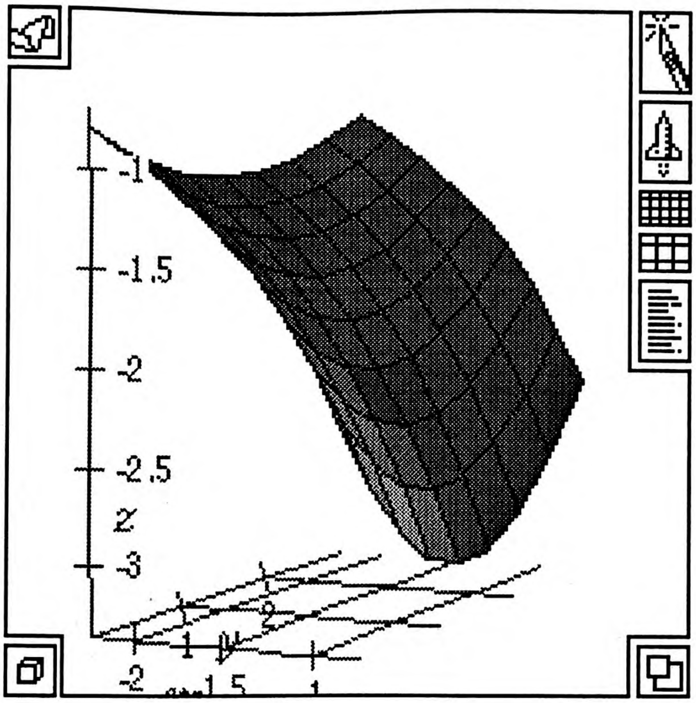
Knife Button

Tip: You can zoom in on a graph by holding the option key (Macintosh) or alt and Ctrl keys (Windows) as you click the Rocketship.

The knife, which zooms in on a specific section of the graph to see more detail, works somewhat differently in 3-D than its 2-D counterpart. To see how it works, re-orient the graph so you are looking down at the top of the surface (or up from below). This process works best when you are looking at a surface perpendicularly (although you can play around with it later). Click on the knife, as before. Your cursor turns into a knife, ready to slice out and enlarge a piece of the graph. This time, you slice out a section of the graph by dragging on the surface itself.



The piece you slice out constitutes a number of whole panels.



Three-dimensional graphs, like 2-D graphs, are interactive and always reflect current conditions. When you change the equation, the graph changes in response.

THREE-DIMENSIONAL GRAPH DETAILS

Each graph is defined by a set of statements that describe the contents of the graph. When you edit a graph, you change certain values in these statements. You can also edit these statements directly. This tutorial provides a brief tour of the graph details for a three-dimensional graph, including:

- Graph bounds
- Axes propositions
- The aspect ratio
- Display cropping
- Viewing lens

In a new notebook, make an Illuminated 3-D graph with the equation

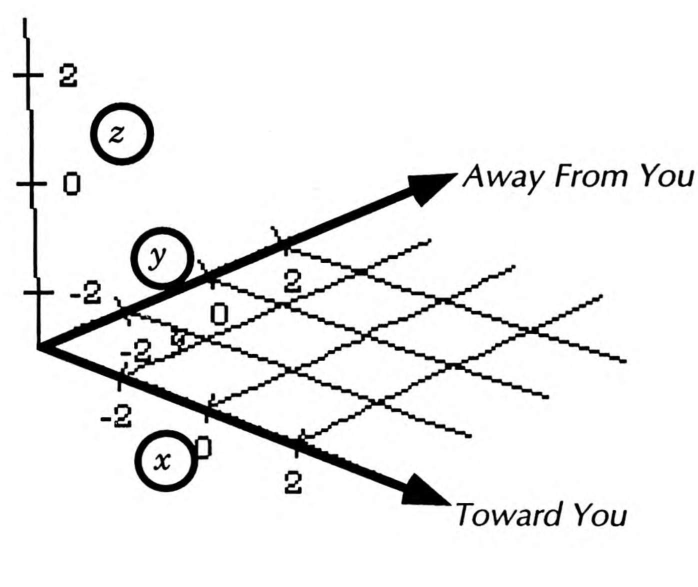
$$z = \frac{x + y}{2}$$

Surface Plot Icon

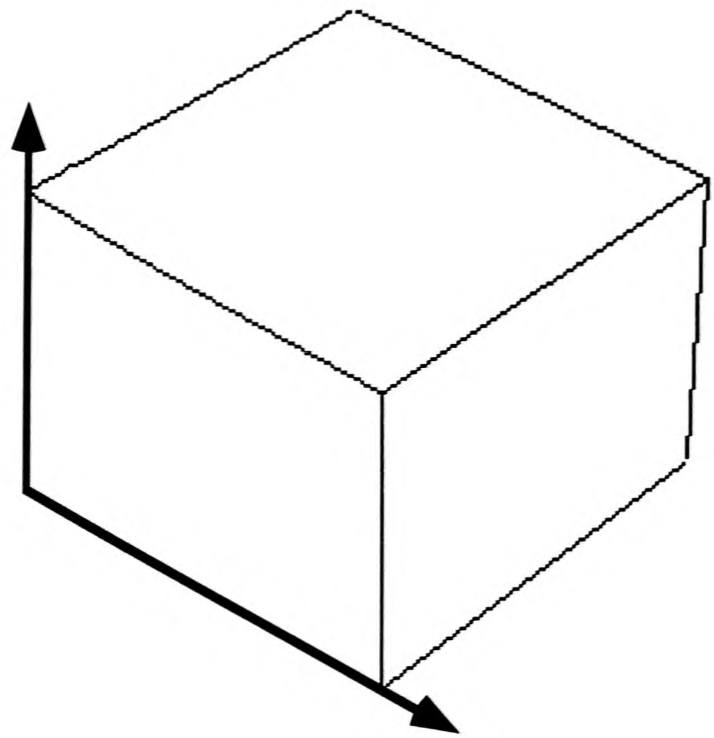
Open the graph details and delete the surface plot, then close the details. Delete the entire equation proposition because all we need is an empty 3-D graph.

Right Hand (palm up)

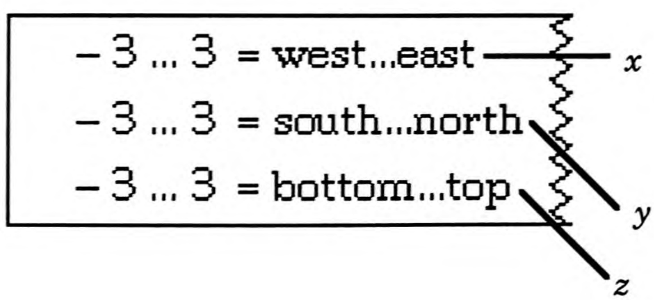
Note the coordinate axes. You have a right-handed coordinate system. If you take your right hand and label your thumb and forefingers *x, y,* and *z* in order, you get the correct orientation when you spread your three fingers apart like a tripod. Left-handed coordinate systems are mirror images of these.



Rotate the graph with the hand. Note that the axes all line up with the ghost cube.



The cube outlines a piece of three-dimensional space called the data region for that graph. The edges of the data region are the bounds. You can see and edit the bounds by opening up the details of the graph.



Since 3-D graphs are designed to work for all variables, not just *x, y,* and *z,* these bounds are referred to by geographic names. If you orient the axes on end, you notice that the west, south, et cetera all work as on a map or on the surface of the earth, with the top higher in altitude than the bottom. Note that the south, west, and bottom numbers are always lower and that the north, east and top numbers are always higher. Remember this. Think of Los Angeles (southwest) as being negative and New York (northeast) as being positive.

Expose and examine the Declarations comment within the graph details. Beneath it are six name declarations, one for each of the bounds numbers.

A Constant▼ named west behaves as 3D X-Minimum▼.
A Constant▼ named east behaves as 3D X-Maximum▼.
A Constant▼ named south behaves as 3D Y-Minimum▼.
A Constant▼ named north behaves as 3D Y-Maximum▼.
A Constant▼ named bottom behaves as Vertical Minimum▼.
A Constant▼ named top behaves as Vertical Maximum▼.

Within the graph these names behave as constants that always evaluate to the bound value of their name.

Also under the graph declarations are three axis plots, one for each dimension:

Axis at (west, south, z) where z = bottom ... top labeled z on this side▼ colored Lilac▼.
Axis at (west, y, bottom) where y = south ... north labeled y on this side▼ colored Lilac▼.
Axis at (x, south, bottom) where x = west ... east labeled x on this side▼ colored Lilac▼.

Each one consists of a vector that would be (west, south, bottom) except one of the components has a variable in its place. The location (west, south, bottom) is the low point within the bounds because each coordinate is at its minimum there. Also, all the axes start here in the current set-up.

Click on the first axis icon to select it, copy, and paste three times. Change half of the souths and bottoms to their opposites so you have all four combinations:

x, south, bottom
x, south, top
x, north, bottom
x, north, top

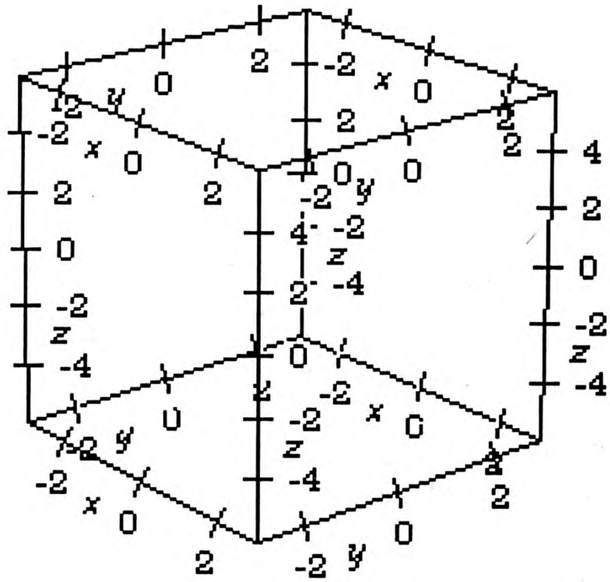
Your graph now has four x-axes. Do the same thing to the y axis:

west, y, bottom
west, y, top
east, y, bottom
east, y, top

…and the *z* axis:

west, south, *z*
west, north, *z*
east, south, *z*
east, north, *z*

Delete the grids, then take a look at the graph. You get a tricky, yet educational, result. You also get a fuzzy cube.



This cube of axes is good because you get a better idea of what is going on in the space.

All plots in your 3-D graph are drawn in the graph space. The viewer can see anything that is visible in the viewport, not only in the data region but beyond it. (For efficiency, Theorist does not draw objects that are more than a few diameters away from the data region.) You can rotate this space around and view any part of the space by typing its coordinates into the bounds.

Stretch to Fit
True Proportions
Aspect Ratio pop-up menu

The aspect ratio controls how Theorist reacts when your bounds are not the same size. To see how this works, change your “bottom…top” bounds from “–3…3” to “–1…1”. Examine the graph as you rotate it with it stretched to fit, then rotate and examine with true proportions.

“Stretch to Fit” means the space might be distorted along one or more axes in order to make the data region appear to be a perfect cube. Angles and slopes are distorted. This setting is best if your dimensions are not measured in the same units (for example, pressure versus temperature).

“True Proportions” means that one unit in the *x* direction corresponds to one unit in the *y* and *z* directions, such as when you are drawing geometric diagrams in the space where *x, y,* and *z* are all in units of length.

As you rotate a truly-proportioned graph, you may get a rectangle because the data region is not truly cubic. But, as you rotate a stretched-to-fit graph, there is always a distortion in effect that causes it to look like a perfect cube. You can see this by watching the tick mark spacing on the axes.

Looking at the graph through the viewport is like looking through the viewfinder of a camera. Theorist makes it easy to control all the options with two pop-up menus, one for cropping and one for viewing:

cropped [Moderately ▾]
as seen through a [Normal ▾] lens

The cropping determines how close the image gets to the edge of the viewport. Since your data can produce strange plots, you can set this on Very Wide, to see everything. (Do this now.) The image is shrunk so it all fits in. This option is particularly useful if your data has singularities or goes far beyond the data region.

On the other hand, you may want to get a more close-up, dramatic view of some small part in the middle. You can get to a desired view in very tight steps if needed. Do this and experiment.

The lens setting determines the degree of perspective. The lowest setting is Very Wide Angle, which almost always distorts a lot. Choose this now. The main distortion is that close objects get very large. Therefore, widen the cropping at the same time.

Distortion lessens toward the other end. The last few are barely distinguishable on the screen but are visible on high resolution output devices such as laser printers. Try Telephoto now.

The last setting, Infinitely Distant, removes all perspective distortion. All parallel lines remain parallel. Try it now.

You can use the options described in this tutorial to customize your graph view to your exact needs.

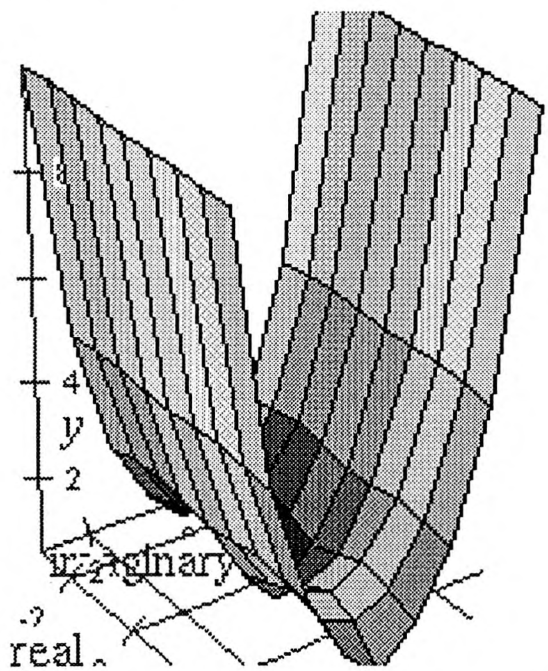
Complex 3-D Graph

If your machine is equipped with color, you can use color as an additional dimension to tell you more about a relationship. (Even if you only have black and white, you can use Theorist's shading.)

One way this comes up is in functions of a complex variable. Each complex dimension soaks up two real-valued dimensions. Therefore, a simple complex relationship $y = \mathrm{f}(x)$ results in a four dimensional graph! Theorist has a way to deal with this.

Make a new assumption and type **y=sin(x)**, a function which works all over the complex plane. You can witness firsthand what it looks like. With a selection anywhere in the equation, pull down the **Graph** menu and choose **Complex 3D** from the **y = f(x)** submenu. (Ignore the first two axis variables in the dialog.)

You may want to resize and/or move the graph to get a better view.



The complex 3-D graph uses the absolute value of the function for the height and the phase to indicate the hue. As you travel around the circle $|x| = 1$ in the complex plane, the color rotates around the rainbow. As the magnitude increases, the color bleaches to white, and as the magnitude is reduced, it fades to black. It is more spectacular in color than in black and white, because the color indicates the complex phase of the function.

This can be seen graphically in the plot. The absolute value of $\sin(x)$ along the real axis proceeds as it does for real-valued $\sin(x)$, but along the imaginary direction, it turns hyperbolic:

$$\sin(ix) = i \sinh(x)$$

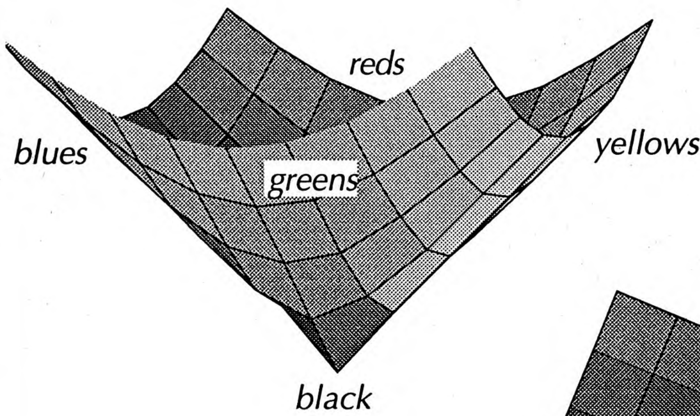
If all the above seems confusing to you, follow the next tutorial and then perhaps you can see how the complex 3-D graph does what it does.

Surface Coloring Schemes

Color is a useful additional dimension to tell you more about a relationship. (Even if you only have black and white, you can use Theorist's shading.) This tutorial describes different coloring schemes used with surface plots in three-dimensional graphs.

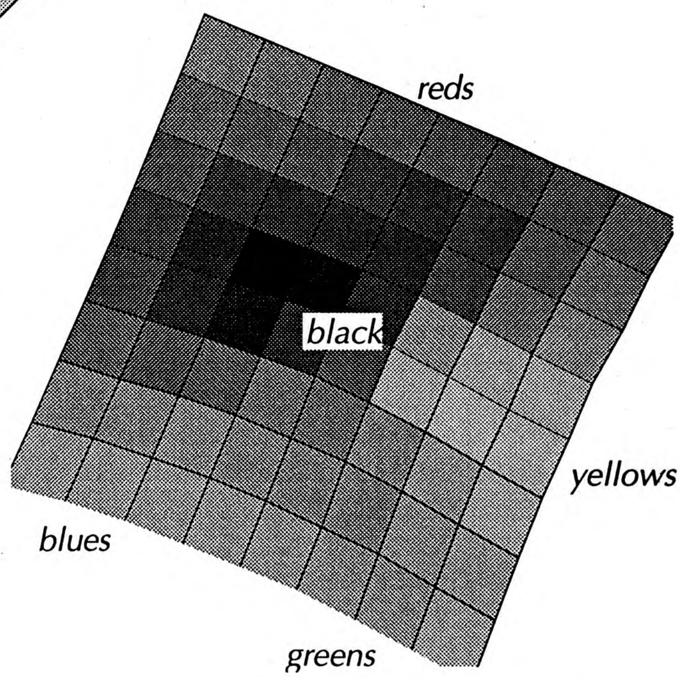
Make a new assumption and type **y=x**. With a selection anywhere in the equation, pull down the **Graph** menu and choose **Complex 3D** from the **y = f(x)** submenu. (Ignore the first two parameters in the dialog and match up the *x* and *y* axis variables.)

You may want to resize and/or move the graph to get a better view.

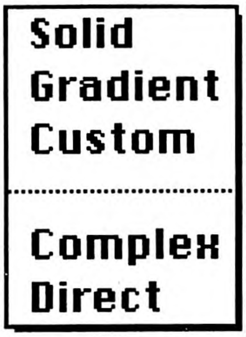


▲ *Side View*

Top View ▶



Open up the graph details and look at the surface plot. The **surface plot** is the most complex of the five available kinds of plots because of all the coloring options.



Coloring Technique pop-up menu

The master switch for the color is a pop-up menu, set to Complex below.

There are five coloring options. Many of them use the value from the evaluated expression. In this case, Complex, the surface plot expects this expression to be complex-valued. If it was only real-valued, your colors would not be as interesting.

Change this pop-up menu to **Gradient**. When the graph is redrawn, surprise—only the mesh is left. This is because Gradient expects the shade/color expression to be real-valued and it is not. More specifically, it expects it to be in the range 0...1. Each pane for which the expression is not real, Theorist makes transparent. To fix this, change the complex expression (which appears just before the words "selects a color from") from y to $\left|\frac{y}{5}\right|$.

$\left|\frac{y}{5}\right|$ selects a color from [Black ▾] ... [White ▾].

As the name implies, the color is a mixture of the two selected colors blended from 0 to 1. You can set an alternate minimum or maximum. Change the expression to $|y| = 1...3$. In this case, 1 becomes the minimum and 3 becomes the maximum.

$|y| = 1 \ldots 3$ selects a color from [Black ▾] ... [White ▾].

Optical Properties pop-up menu

An even simpler color scheme is Solid. This option lets you choose a color which is constant over the surface. It is most useful if you choose Illuminated from the first pop-up menu, which chooses optical properties of the surface.

Choose the **Custom** coloring option. This option is best when you need to discern levels of intensity. The graph is redrawn in white, blue, green, yellow, and black, according to the value of your expression, which, once again, ranges from 0 to 1 (or other limits you specify).

The complete mapping is:

Value	Color
0	Black
1/4	Blue
1/2	Green
3/4	Yellow
1	White

Colors in between are blended and/or interpolated. Colors beyond are pinned (black or white).

Notice the pop-up menu named Oregon. Custom color schemes all follow the above scheme ranging from 0 to 1, but they differ in the colors they use. The names are states in the United States. Most of them go from black to white through various colors, but Kansas cycles through the rainbow. See the Theorist Reference Manual for a complete listing (or just try them out).

Choose **Direct**. This is a more serious option, where you want to directly calculate the color to use. This scheme requires you to enter a 3-vector into an expression. Change the $|y| = 1...3$ to Re(x), Im(x), 0.5:

(Re[x], Im[x], 0.5) determines the color components [Red, Green, Blue ▼].

Your surface is bathed in subtle shades. Maybe. You may have to use the knife to select such portions of the surface. See the Theorist Reference Manual for more details on this mapping scheme.

Plotting Multiple Surfaces in a 3-D Graph

In a new notebook enter the following equations:

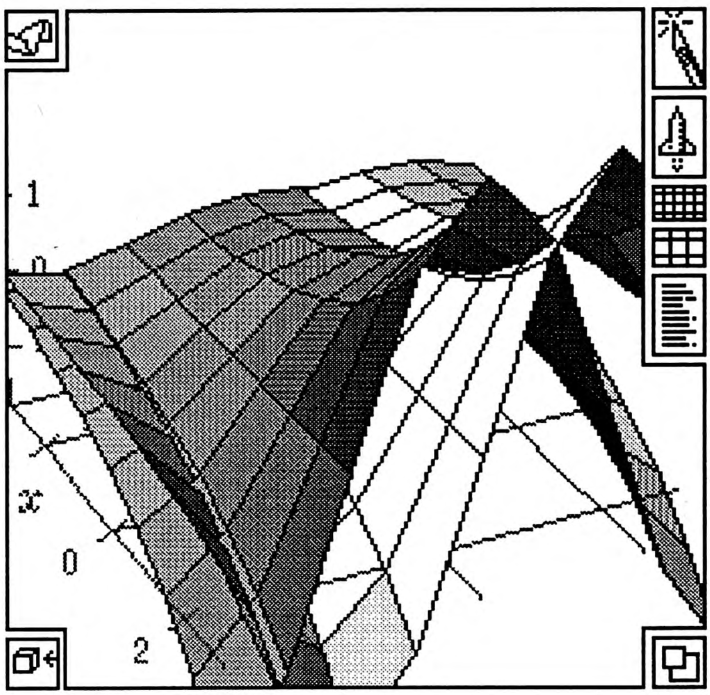
$$z = 1.5^x \sin(y)$$

$$z' = 1.5^x \cos(y)$$

When you get a dialog, just accept the defaults.

Clarify and declare z' as a variable. Click within the first equation and create an Illuminated 3-D graph, then click in the second equation and choose **Graph▸Additional▸Add Surface Plot**.

Rotating the graph helps visualization. (If you are using a color monitor, it also helps if you open the details and make the second plot translucent, possibly with a different color scheme.) Note the intersections of the surfaces.



Parametric Plotting in Three Dimensions

This chapter shows you how to make a space plot of a curved line suspended in space. Graphs of this type are helpful when you are trying to trace the trajectory of a particle in space.

Create a new notebook with the following three propositions.

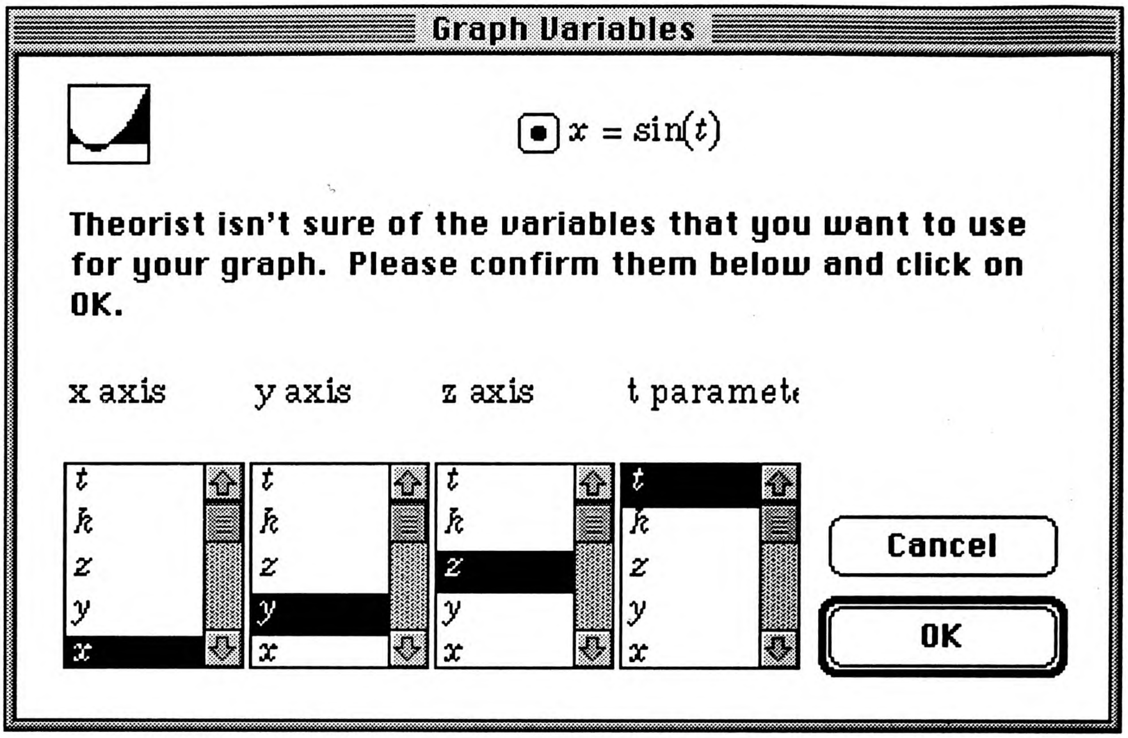
$$x = \sin(t)$$
$$y = \cos(t)$$
$$z = t$$

It might be easiest to click on the proposition icons of each.

Select all three equations, selecting the x one first, then y, then z. (You need not select the whole equation, just click anywhere inside the first one, then hold [shift] as you click on or in the other two.) All three propositions should now be highlighted or should have a selection in it. If you did not select them in order (x first, then y, then z), un-select and go back and do it right.

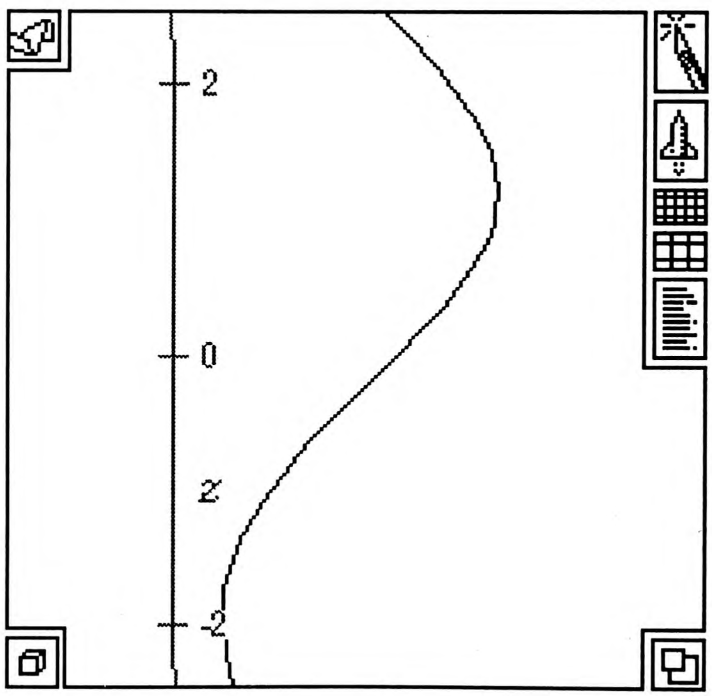
$$x = \sin(t)$$
$$y = \cos(t)$$
$$z = t$$

To plot the helix, choose **x = f(t), y = g(t), z = h(t) Space Curve** from the **Graph** menu's **Other** submenu. (The declaration dialog for t comes up first; make t a variable.) The familiar axis variables dialog comes up.

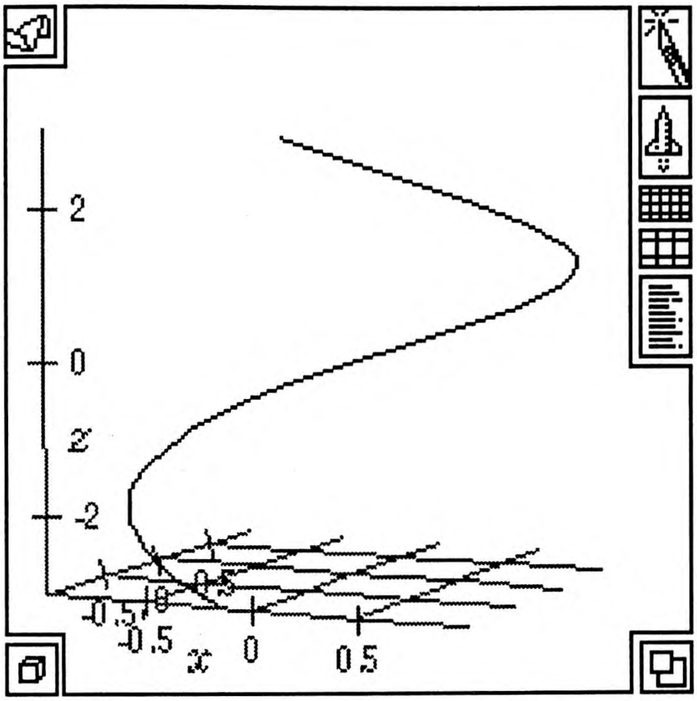


However, it is *not* exactly the same as the dialog we have been using for the past few tutorials; this one has more variables because we are working in three dimensions now. It allows you to change the variables you want to use to plot the curve. If you have done all of the above steps correctly, the dialog shows x, y, and z in the left boxes and t in the right box. If not, simply select them in that order. (Theorist reads things off left to right. If you have other variables with values defined outside the statements, you should select the true dependent and independent variables at this time. Take care when creating graphs that use more than one variable.) Click **OK**.

If you started with a fresh notebook you will not have extraneous variables interfering in this dialog.



It is so tall and skinny all we can see is its middle. Open the graph details and change True Proportions to Stretch to Fit. This changes the graph to something more attractive.



GRAPHING FROM SCRATCH

To create a non-standard graph, such as a three-dimensional Line plot, start by creating the type of graph theory you want as a shell, then add plots as needed

In this example we create a Line plot spiral in three dimensional space.

Enter an equation in three variables, such as $z = x + y$. Select it and choose any 3D item from the **Graph▸z = f(x, y)** submenu. (Accept whatever default dependency relations are displayed in the dialog.) Open the graph details and delete the Surface plot proposition.

To insert a parametric Line plot into this empty shell, choose **Add Line Plot** from the **Graph** menu's **Additional** submenu. (Match the axis variables in the dialog.) Modify the new line plot proposition as follows:

Line at (x,y,z) where $t = -10 \ldots 10$ with a [normal▾] line, colored [Black▾].

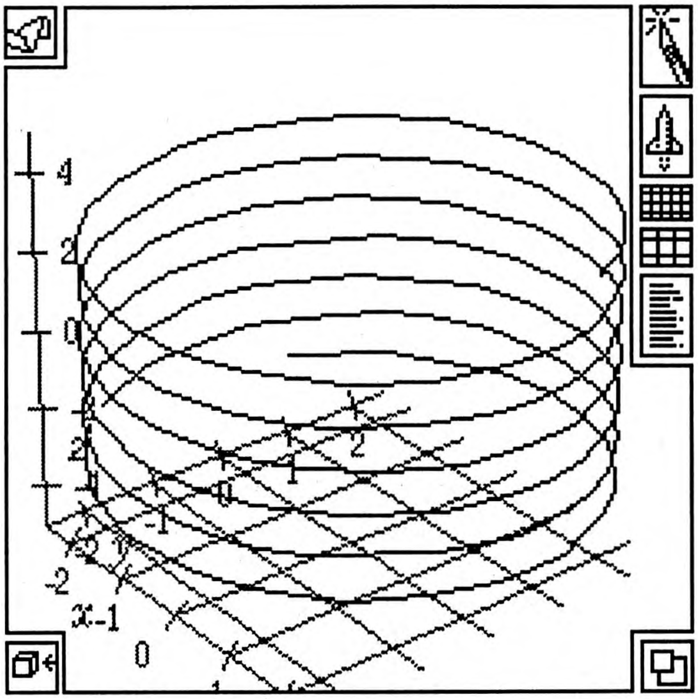
Then, outside the graph, create the following equations, select them, and choose **Make Working Stmt** from the **Notebook** menu. (Declare t as a variable.)

$$x = 3\sin(2t)$$
$$y = 3\cos(2t)$$
$$z = \frac{t}{3}$$

Depending on the equation used to create the three dimensional graph theory, you may need to adjust the graph bounds and the viewport controls. Set the aspect ratio to Stretch to Fit, and adjust the graph bounds to –2.5 ... 2.5 for the two horizontal coordinates (west ... east and south ... north), and –5 to 5 for the vertical coordinate (bottom ... top).

$-2.5 \ldots 2.5$ = west...east	[Stretch to Fit▾]
$-2.5 \ldots 2.5$ = south...north	cropped [Moderately▾]
$-5 \ldots 5$ = bottom...top	as seen through a [Normal▾] lens

With a little rotation, the graph should look like this:



Using this technique, you can insert a Line, Contour, Scatter, or Surface plot into any two- or three-dimensional graph theory.

ANIMATING A GRAPH

Theorist's animation facility lets you view a family of related graphs in quick succession. The animation creates the illusion of movement by creating several versions of a graph as one variable changes, saving these images, and displaying them in rapid succession.

Don't start yet; this is just an overview.

The first step in the animation process is to modify your graph by incorporating an extra "time" or "animation" parameter into some expression. (We will use a as an animation parameter, but you may use any as-yet-unused variable.) This animation parameter changes in value from one frame of animation to the next. Then you give the Start command, which appends the animation box to the bottom of your graph and begins the animation process.

The process of animating even the most complex three dimensional, color graph is basically the same as the process of animating a Saturday morning cartoon. Theorist starts with an image of the graph, then renders many similar images, each drawn slightly different from the previous image. Finally, it quickly flips through all the completed images, or frames, giving the impression of motion.

Because animation is memory-intensive, before you use it you should make sure Theorist has as much memory as possible. (This is major consideration for Macintosh users, but less so for Windows users, because of different memory management and controls.)

Application Memory Size (Macintosh only)

If you are using System 7 or MultiFinder, Theorist can only work with the amount of memory allocated to it. Even if you have a sixteen-megabyte Macintosh, if you allocated Theorist a 1500K partition, you don't have much to work with. To increase the Theorist application memory size, quit Theorist and go to the Finder. Select the application by clicking once on its icon, then choose **Get Info** from the **File** menu to bring up the Info window. The box in the lower right hand corner specifies the memory to be used by the application. Enter the desired memory where it says "Current size" or "Preferred size". For additional instructions, see the Getting Started section's Macintosh installation chapter.

Animation Steps

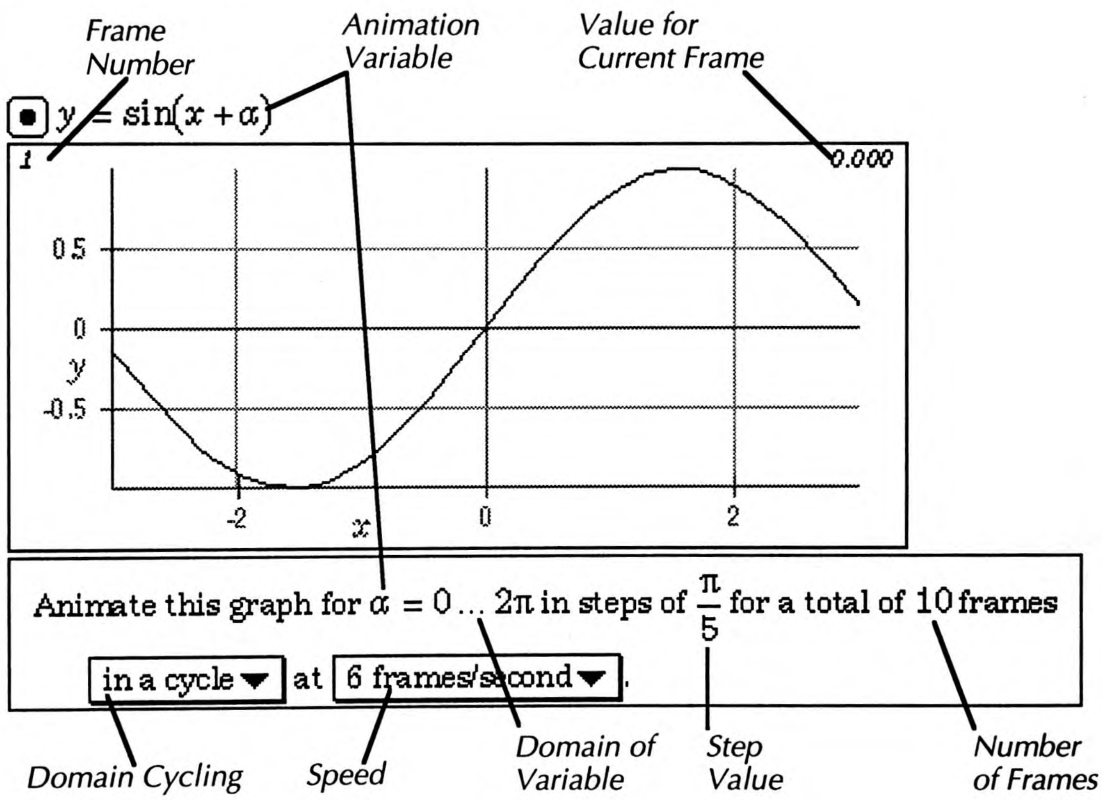
Let's start by animating a simple two-dimensional graph. Create a new notebook and type **y=sin(x)**. Graph it by clicking the graph icon from the palette (or by choosing **Graph ▸y = f(x)▸Linear**).

This reduction is not necessary if you are confident Theorist has plenty of memory to work with.

Reduce your graph to about half its original size by dragging the resize box. This saves memory and more or less guarantees success.

Next we add the animation parameter. Select the x and type **+a**. (After typing + a, an error message appears in the graph area. This is normal and will be taken care of later.) The added variable will cause the phase of the sine wave to vary from one frame to the next when the animation begins.

The next step is to select the animation parameter, so drag over a. Pull down the **Graph** menu's **Animation** submenu and choose **Start**. The animation appears in place of the graph proposition. The animation begins at a start value and increments by the step value for each frame until it reaches the ending value. The first time through the sequence, Theorist draws the graph slowly and carefully on the screen for each frame, saving frame images in memory as it goes along. (This is the initial calculating and rendering; note the appearance of the Calculator icon.) After the last frame has been saved in this manner, the actual animation begins.



To halt the animation click the mouse anywhere in the notebook. To resume, click within the graph viewport. To stop the animation and return the viewport to the inanimate graph state, choose **Graph▸Animation▸Stop**. This stops animation, throws away the rendered frames currently in memory, and gets rid of the animation controls box below the graph.

The size of the animation depends upon the size of your graph. A copy of the viewport is displayed, but without the editing icons. You can not edit a graph while animating it.

Animation Controls

Domain

Below the animation are the animation controls, including a statement with some settings and pop-up menus.

The statement says "Animate this graph for" followed by the animation variable and the domain over which the animation variable will vary. This domain often defaults to run from zero up to 2π (6.28). You can change either or both of these values.

Domain Cycling

Functions that are naturally periodic should use the "in a cycle" setting. Images for the starting and ending values appear the same; you only want one frame for this value, not two. If your function is non-periodic, use the "one-way" setting which pauses briefly between the last and first frame.

Number of Frames

More frames also equals more time to render them.

Following the domain in the statement is the total number of frames to use in the animation. It's simple: more frames equals more quality equals more memory usage. More frames produce smoother animations. Available memory limits the actual number of frames that you can create. The number of frames to produce defaults to ten (regardless of the available memory). If more memory is available, you can increase the number of frames. If not enough memory is available, decrease the number of frames. (Otherwise frames are generated during the animation, which is slow and undesirable.)

Step Value

Increasing the number of frames decreases the step value and improves the visual resolution of the animation. The step value is determined by dividing the range of values by the number of frames. Changing the starting or ending domain values (or both) adjusts the step size but does not change the number of frames. Changing the step size directly does affect the number of frames.

Speed

For comparison, most cinematic movies are displayed using thirty frames per second.

Use the speed pop-up menu to set the speed of the animation playback. Speeds range from one frame every two seconds (1/2 fps), to thirty frames per second. One additional setting, "Maximum," displays the images as fast as your machine can move them from memory to the screen. These settings are approximate. If the images are very large, or you are running Theorist on a slower CPU, the time required to move the image from memory to screen may be greater than the designated increment between frames.

Tables

Introduction

A table is a model of a smooth, continuous function defined by function values at regular intervals. It is the computerized equivalent of the printed look-up tables of logarithms or trigonometric functions that were in common use before pocket calculators incorporated these functions. With Theorist, a table for any kind of function can be easily generated. Alternatively, you can create your own tables by importing data from an external source or entering the numbers by hand.

Theorist's tables are much more efficient at storing large quantities of numbers than expressions such as matrices. Whereas matrices of over 100 rows can get unwieldy, tables can easily handle tens of thousands of entries. Memory size is the only limit of table size.

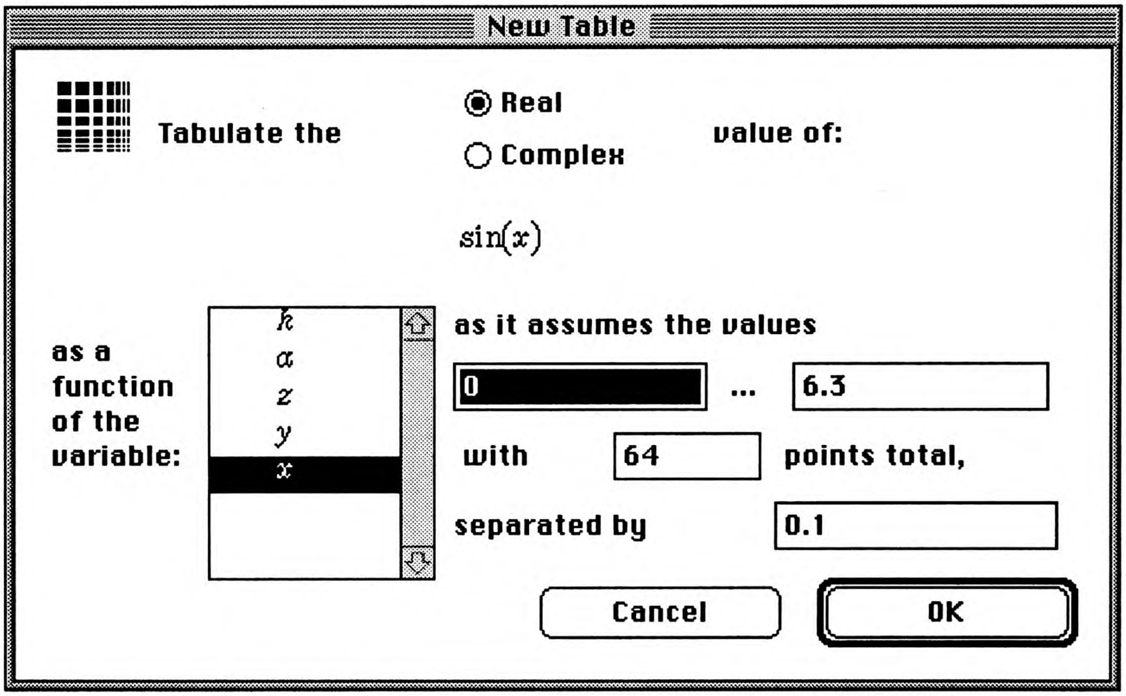
You can create a table from an expression, a graph, external data, or by integrating differential equations. Examples of how to create a table using each method are described in the next few chapters.

TABLE BASICS

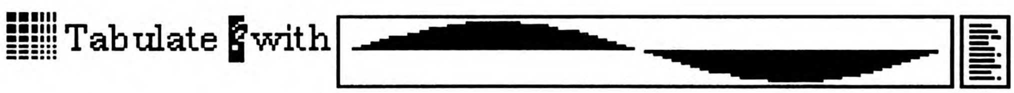
This tutorial shows:

- how to generate a table
- how to draw a graph of a table
- some table properties

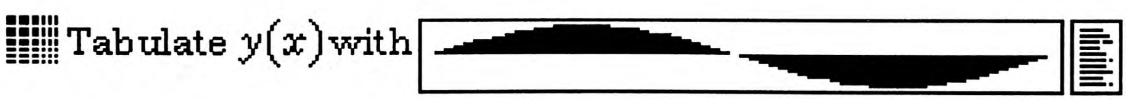
Create a new notebook and type **sin(x)**. Select it and click the palette button (or choose **Manipulate▸Table▸Generate...**). This brings up the New Table dialog.



Click **OK** to accept the dialog defaults. Theorist inserts the table proposition in the notebook.

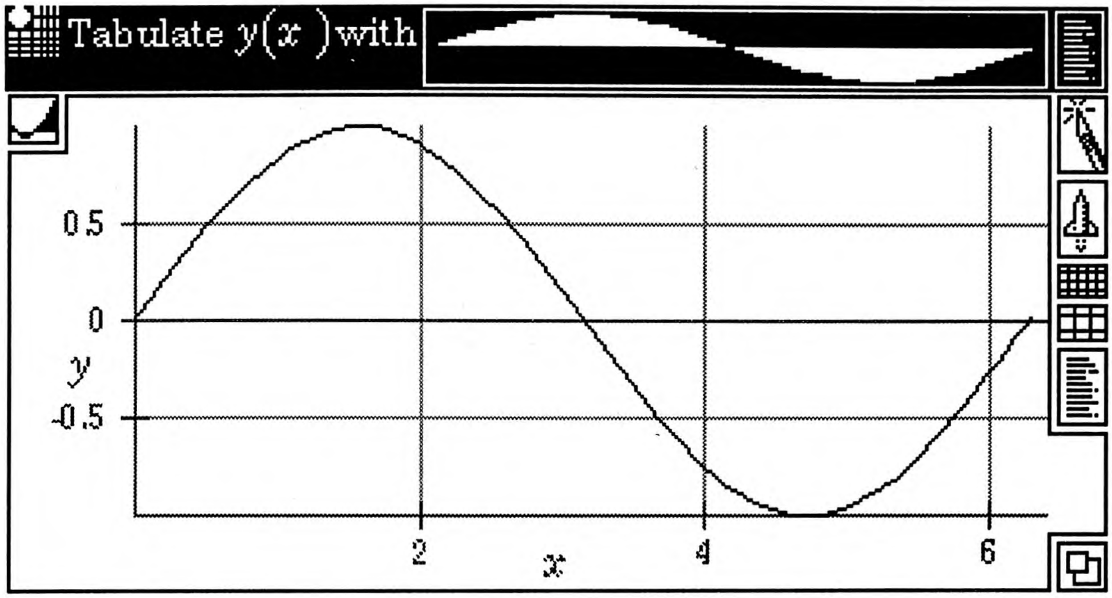


With the ? in the table proposition selected, type **y(x)** for the table name.

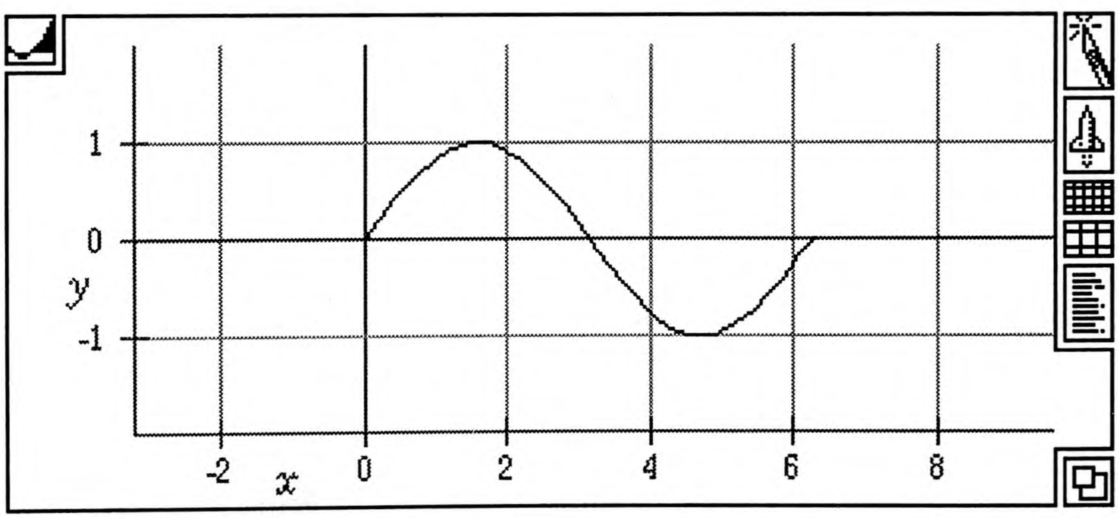


This tells Theorist to use this table to find values of y with respect to x, similar to if you had typed in $y = \sin(x)$.

The value of y may now be graphed in the usual manner. Click the proposition icon to select the table, then click or choose **Graph▸ y = f(x)▸Linear**.

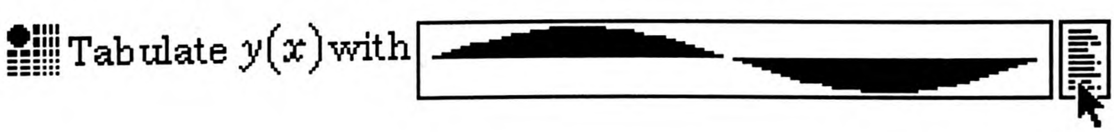


This looks just like $\sin(x)$ but in fact its domain is limited. Click on the Rocketship button to zoom out. Notice that beyond the table's domain, the function is undefined.

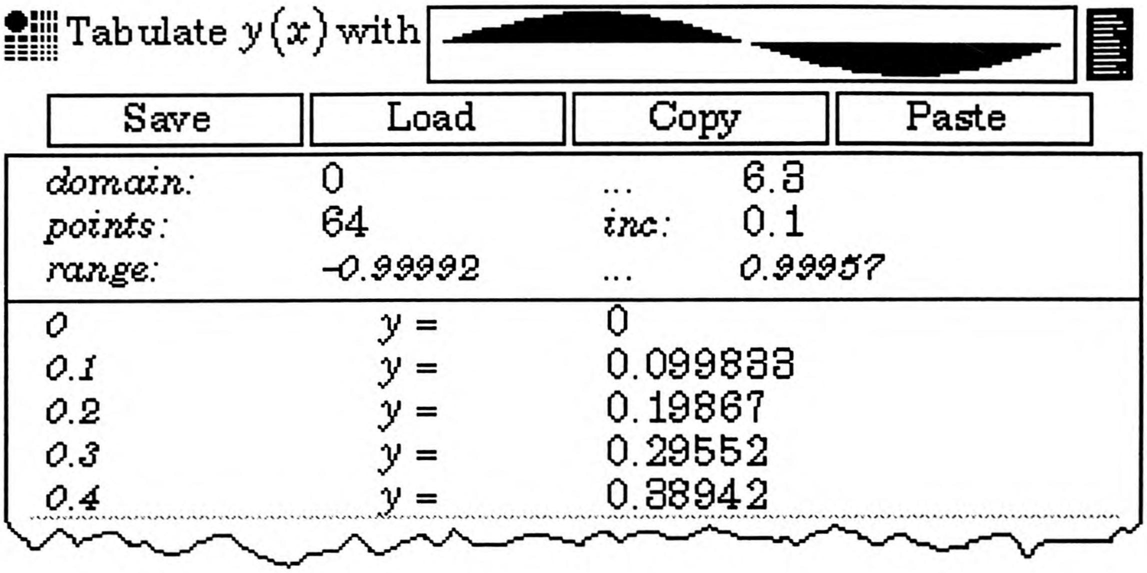


Theorist interpolates between table values to give you a smooth result no matter how coarse your data values are.

Click the icon on the right of the table proposition, like this:

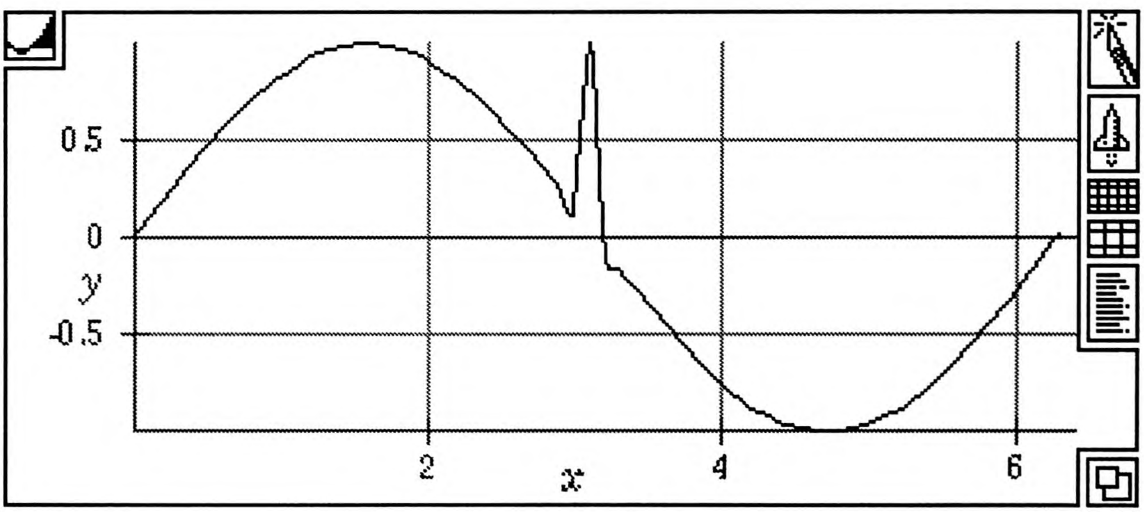


This opens the table details below the table proposition.

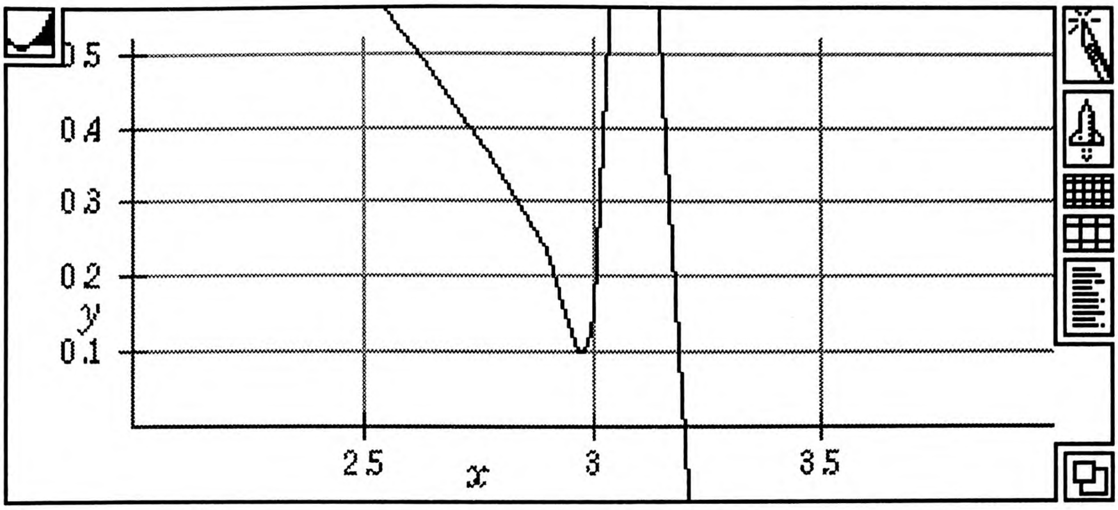


The top area of the table details has buttons for importing and exporting data. Below these are the table domain settings. Below these is along list of data values in the table, with x values at left and y values at right.

In the data, find a value near 3.1 or so (it should be close to zero) and change its value to 1 to try to throw Theorist off. Close the table details and look at the graph. Zoom in and notice the anomaly you introduced by typing 1.



Using the Knife, zoom in on that location to see it in more detail.

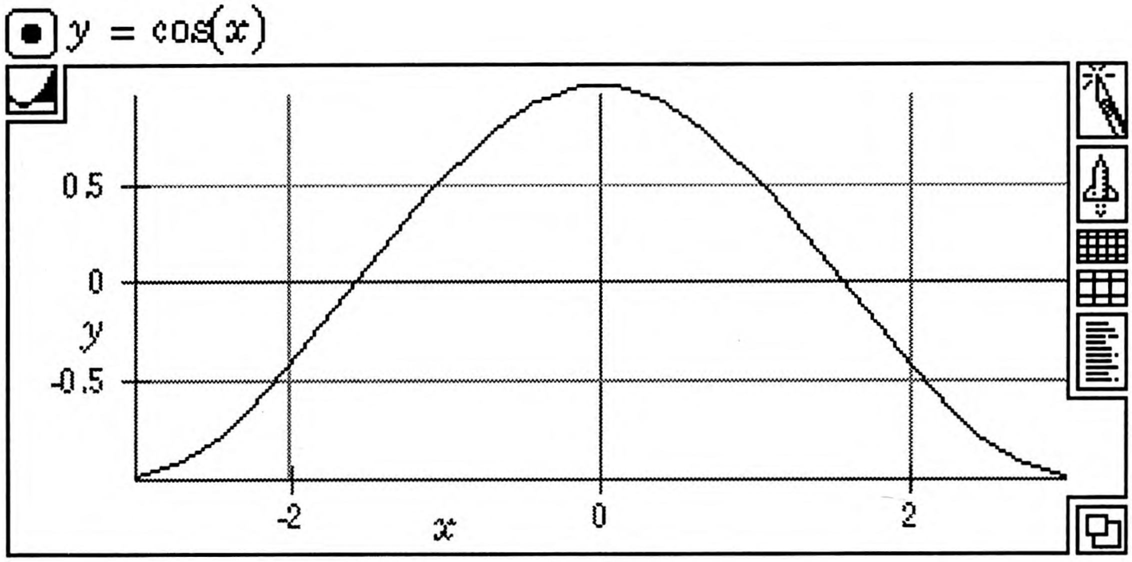


Theorist uses a cubic interpolation algorithm to figure out values between the table values. The algorithm guarantees the resulting function passes through all data values, and the function and its first derivative are continuous, so the result is a smooth, curvy line, despite eccentric data values. Notice that the value takes a swing downward just before reaching the peak just like a large truck going around a sharp turn in a road. This shows how a particular location is affected by its neighboring four values.

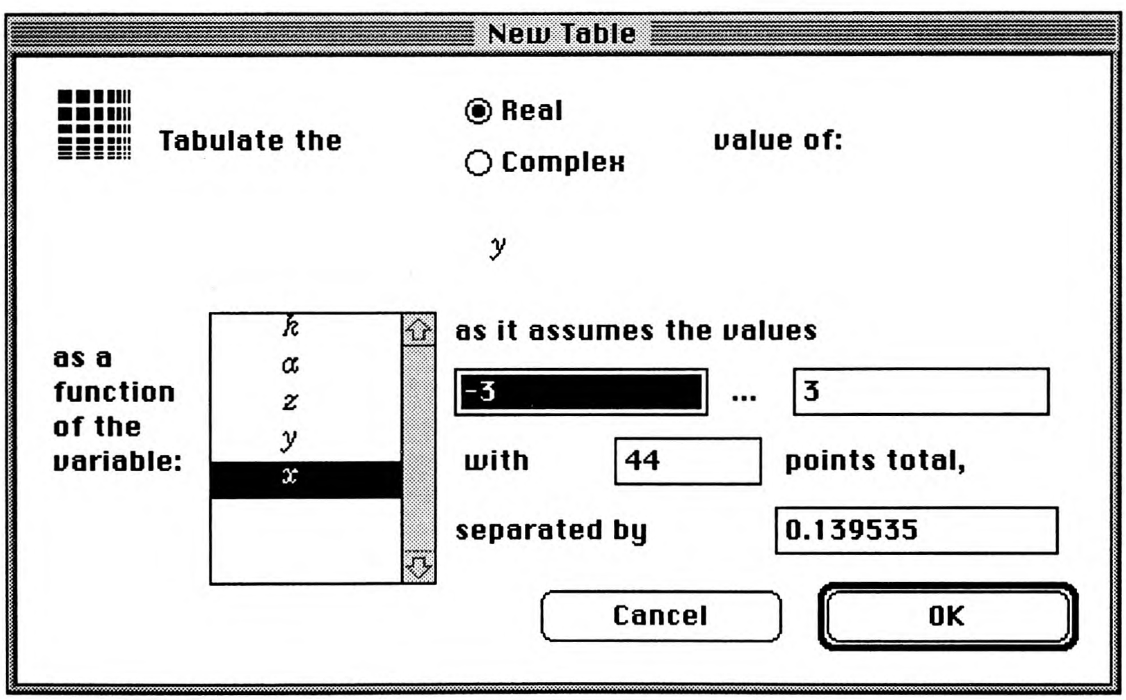
It is also possible to define a function with a table. See the "Graphing the Imaginary Part of a Complex Table" which follows in this section of the manual.

CREATING A TABLE FROM A GRAPH

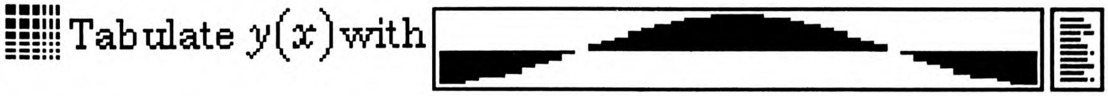
A table may be created from a graph very easily. Create a simple linear graph of any function:



Select the graph (by clicking on the graph proposition icon), then choose **Manipulate▸Table▸Generate** from the menu bar. The New Table dialog appears.



The variable to evaluate and the domain are taken from the graph. If you like, you may change these, as well as the number of data points to store, and then click **OK**.



Note the similarity between the original graph and the snapshot display of the tabular data. If you typed different parameters in the dialog, note the differences between the graph and the snapshot.

If you did not have a graph selected, Theorist looks around for the first graph it finds, and then looks for the first line plot in that graph.

If a particular graph has more than one line plot, you can open the graph's details and directly select that line plot before giving the generate table command. The last component of the vector will be tabulated.

Creating a Table from External Data

You may need to examine data created by another program or gathered from an external source, such as a spreadsheet application. Tables allow you to import data from such sources.

First, create an empty table by clicking the palette button. (Or choose **Manipulate▸Table▸Generate...**.) Just click **OK** at the dialog; the settings are not important because everything will be replaced.

Go to your spreadsheet program. (A word processor will do if you do not have a spreadsheet.) In a new worksheet, type in random numbers, pressing return after each. (The numeric keypad is good for this.) Type in a dozen numbers or so, then select the whole column by clicking and dragging. Choose **Copy** from the program's **Edit** menu. Go back to Theorist and open the table details and click on the **Paste** button in the table. (Do *not* choose Paste from the Edit menu or use the command keystroke.) Theorist adjusts the length of the table to fit the data.

You may experience a glitch depending upon how your spreadsheet program is displaying the numbers. When you copy the numbers out, if there are commas separating thousands, Theorist interprets the number as two or more numbers. For instance, the text "12,345" will be interpreted as two numbers, "12" and "345". By setting formatting options in your spreadsheet, you can avoid this problem.

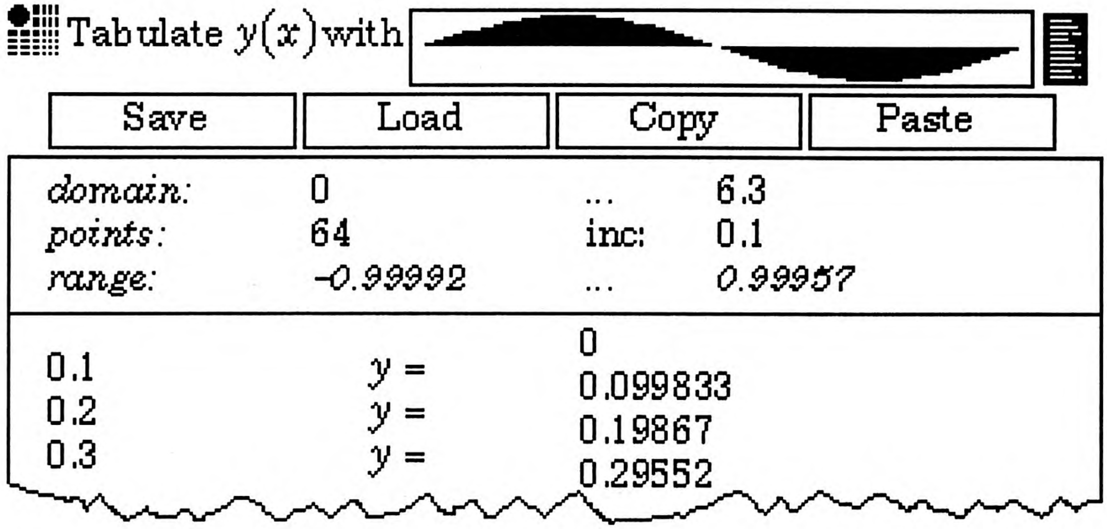
Go back to your spreadsheet program and save the worksheet as a text file. Go back to Theorist and your table. Click on the **Load** button and use the file selection dialog to open the file you just saved. This button lets Theorist read in a text file that has numbers in it. The syntax rules for the text file are exactly the same as for the Paste data.

After you read in your data, you may want to adjust the domain to reflect which data value belongs where on the x axis. If you have two-column data, and the numbers in the first column are equally spaced, you can Paste or Load that into a table and Theorist adapts the domain to fit your data. If the numbers are not equally spaced, Theorist changes the domain to be simple integers.

Reading data into a complex-valued table is similar to that for a real-valued table, except that instead of one or two columns of data, you must have two or three columns of data; the extra column is the imaginary part.

EXPORTING DATA FROM A TABLE

Create a table of data using one of the methods described in the previous tutorials. Open the table details.



Click on the (Save) button, which brings up a file selection dialog. This dialog allows you to give a file name and location where the data from the Theorist table will be stored. After it has been saved, the data can be imported back into Theorist at a later time or imported into one of the many popular data presentation programs. The data for a real table is always a single column of numbers, separated by new lines. For complex tables, it is two columns, for the real and imaginary parts.

You can also transfer the data from a table to the clipboard. Simply click on the (Copy) button.

GRAPHING THE IMAGINARY PART OF A COMPLEX TABLE

In this example, we see how to retrieve a particular column of data from a complex table and how to reference a table by function name.

In a new notebook, enter e^{ix}. Select it and generate a table. The resulting table looks similar to the ones you have made before, but it is in rainbow colors.

☐ e^{ix}

This tells you it is a complex table. The height tells you the real part and the color tells you the complex phase.

Color	**Complex Phase**
yellow	*real positive*
blue	*real negative*
strawberry red	*imaginary positive*
sea green	*imaginary negative*

Next, type **T** over the selected question mark in the table proposition. Clarify and declare T as a function. This is a different way to access a table—as a function instead of as a variable. Note that you do not have to enter $T(x)$ here. You instead specify which variable (x) when you use T, not when it is defined in this table. (Compare this to the variable method where you type $y(x)$ in the table but y when you use it—just the opposite.)

Enter the following proposition.

☐ $y = \mathrm{Im}(T[x])$

Click or choose **Graph▸y = f(x)▸Linear**. (Declare Im as the predefined function.) The resulting graph looks like the ubiquitous sine wave. This should not be too much of a surprise if we recall that $e^{ix} = \cos(x) + i\sin(x)$.

The real part (first column) of a complex table may be graphed in a similar manner with $y' = \mathrm{Re}(T[x])$.

GENERATING RANDOM NUMBERS

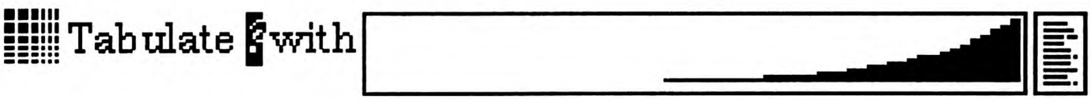
Random numbers are hard to generate with Theorist, because Theorist is deterministic. It is designed specifically to get the same answer every time, whereas a random number generator generates different numbers each time. Nevertheless you can generate a table of pseudo-random numbers that all depend upon the *x* value in a table.

Type **mod(exp(x+3*'p),1)**.

$$\square\ \mathrm{mod}(\exp[x+3\pi],1)$$

Select it and generate a table using the dialog's default settings. Theorist makes a table of pseudo-random numbers, each falling within the range from 0 to 1.

Let's examine how it works. Select just the "exp[*x* + 3*p*]" part of the expression and make a new table.



Open the table and look at the results. You see a typical exponential climb of numbers. All the mod() function does is chop off the whole number part, leaving fractional parts that are scattered widely and evenly over the desired range.

You can customize this same algorithm for your purposes if you put some thought into it. You can change it to 1…2 by simply adding 1. You can change the range to 0…5 by simply multiplying the whole thing by 5.

You might think that you could change the range to 0…5 by changing the 1 to a 5, and that works in this case, but if you try numbers that are much larger, your random number generator will suffer. This can also happen if you change your domain to be numbers very different from 0 through 6.3 with a 0.1 step size.

To see why, try using this random number generator: mod(exp[*x*],1).

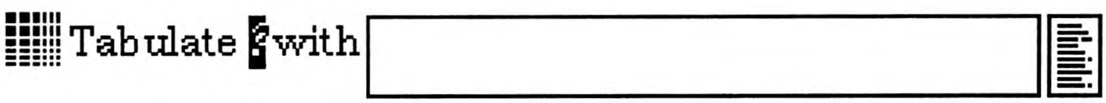
The table, at first glance, looks random, but if you look at the beginning, you will see a ramp of about seven or eight points. Right after it is a sharper ramp, about half as many points, and after that, more ramps going up and down.

Generate a table of the inner function (exp[x]) to see what is going on. The function varies slowly, and the first few ramps are actually pieces of exponential curve. The following ramps are harmonic patterns: if the function is increasing by 4.1 each time, the modulus is changing by .1 each time and you have a ramp.

In order to have a good random number generator, you need the numerical equivalent of a roulette wheel that spins perhaps a hundred or a million times before it stops again, and the exact place where it stops determines the next random number. Spinning a few or a dozen times is not good enough, because lower numbers become more probable than higher numbers.

Note also that the choice of exp() was made to minimize patterns. If you instead use a quadratic in its place, you will also find patterns, but those do not go away so easily. You can experiment with other functions, but avoid those that have smooth minima or maxima; use a monotonic function. Each minimum or maximum results in an unfair proportion of numbers in one local area. (This is why we use mod() instead of sin(). You do not want a lopsided roulette wheel, do you?)

Also avoid going overboard. For instance, try this random number generator: mod(exp[x + 100],1).



You can see the answers are far from random. If you generate a table of the inner function, you can see why: all of the numbers are greater than 10^{40}. Since Theorist has only 15 digits of precision, any number greater than 10^{15} lose all precision in its fractional part. Numbers that are about 10^{10} only have five digits in their fractional parts. You must keep this in mind while designing your random number generator.

This is a discussion of very simple random number generators that are easy to type into Theorist. A more thorough discussion is in "The Art of Computer Programming," volume 2, Chapter 3, by Donald E. Knuth, Addison Wesley Publishing. (See the Bibliography in the Theorist Reference Manual.)

Algebra

INTRODUCTION TO POLYNOMIALS

This tutorial introduces the concept of nomials and polynomials, and uses Theorist's graphing ability to show the form of the first four "orders" of polynomials.

What is a Nomial?

binomial:	$ax + b$
quadratic polynomial:	$ax^2 + bx + c$
cubic polynomial:	$ax^3 + bx^2 + cx + d$
quartic polynomial:	$ax^4 + \ldots + e$
polynomial:	$ax^n + \ldots + z$

A polynomial is an expression that takes on a special form. The form is a sum of powers of a variable (usually x), each being multiplied by some constant. Here are some examples of polynomials:

$$x^2 + 2x + 1 \qquad 3x^3 + 2x^2 + x + 1$$

$$1 - x^{30} \qquad \frac{7}{3}x^2 + \frac{10}{13}x - \frac{42}{5}$$

Polynomials are very useful because you can get a wide variety of different functions of x by controlling those coefficients.

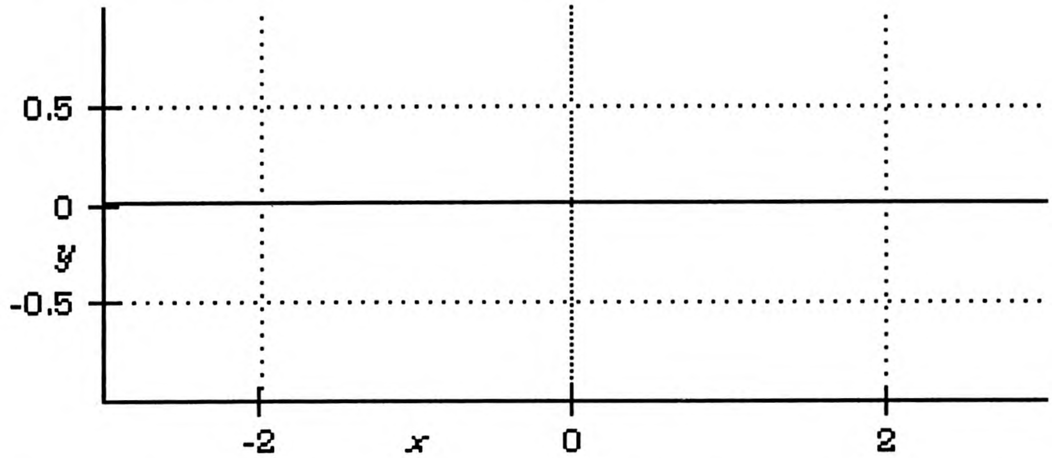
The 'order' of the polynomial is the highest power present. In the four examples above, two are third-order, and two are second-order.

In a new notebook, type **y=(0*x^3)+(0*x^2)+0*x+0**.

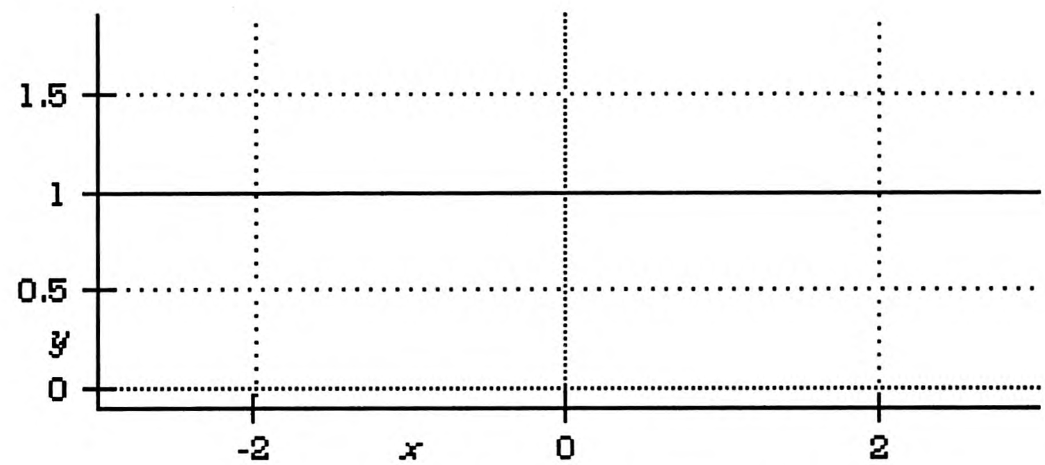
On the screen, Theorist gives you:

$$\square\ y = (0x^3) + (0x^2) + 0x + 0$$

Click anywhere in the equation and choose **Graph▸y = f(x)▸Linear** to get a graph of it. Click the rocket button to zoom out. A graph of zero looks pretty boring.

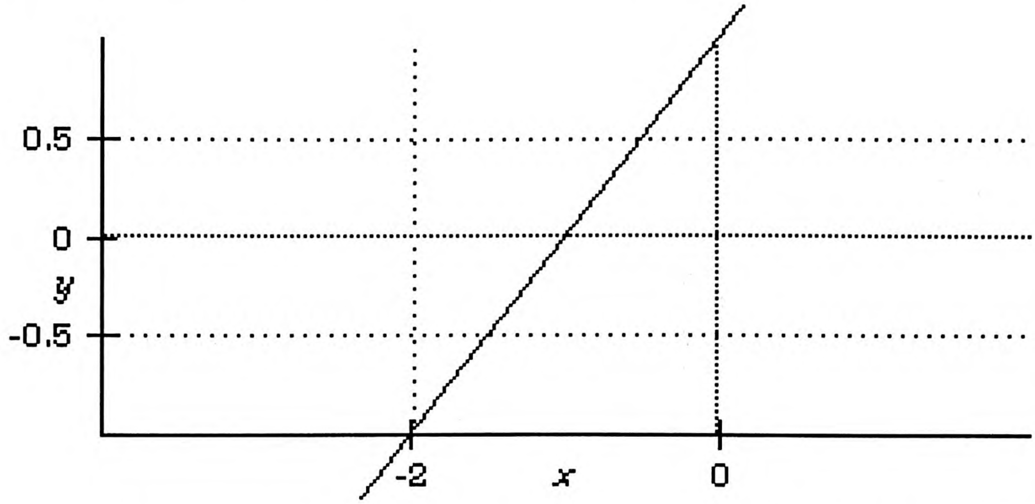


In the equation, select the last zero and type **1**. The graph changes so you are looking at a zero-order polynomial. Use the hand to get a better view.



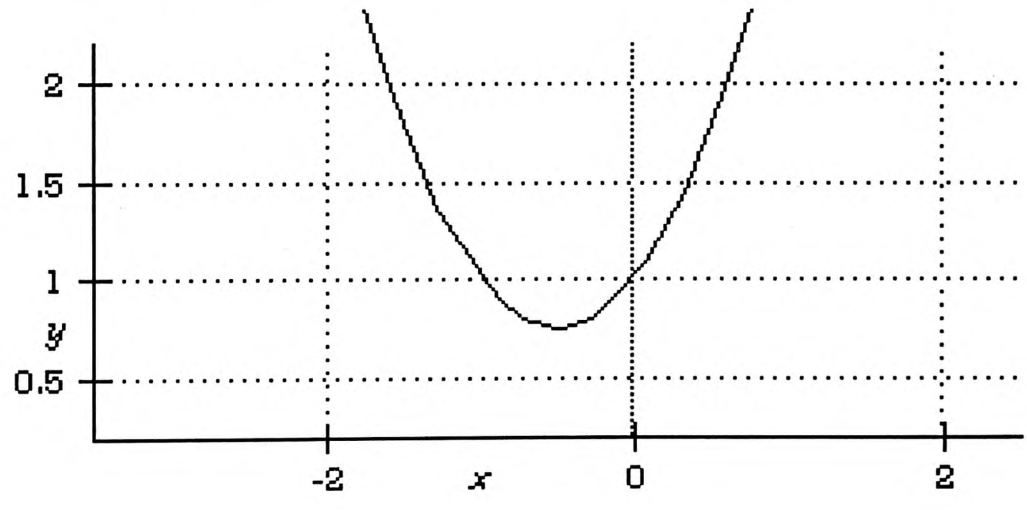
The zero-order polynomial is a constant value. Constant values are used for approximations for many things. They are not very accurate but they are extremely easy to use. By changing the coefficient to different numbers, you can change the height of the function everywhere at once.

Now change the next zero to the left, the $0x$ term, to 1. x really means x^1, just as $x^0 = 1$. Use the hand to center the graph.

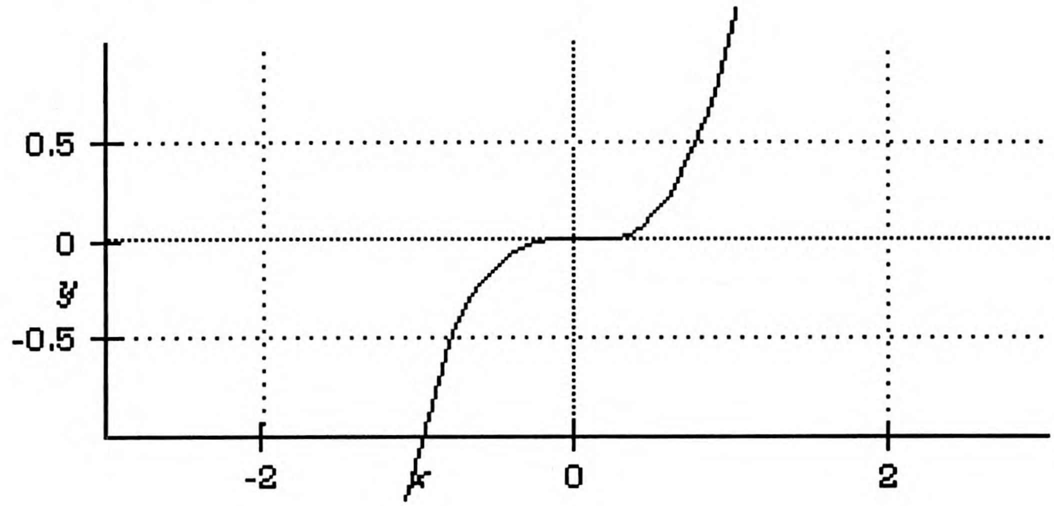


Now the function is a line. Straight lines are great for making approximations. You can change the slope (which is the tangent of the angle) of the graph by changing this first-order coefficient. As before, you can change the height by changing the zero-order coefficient. By varying these two you can get almost any straight line. The only kind you can not get is straight up and down. (Try it.) When you are done experimenting with these two coefficients, set them back to 1.

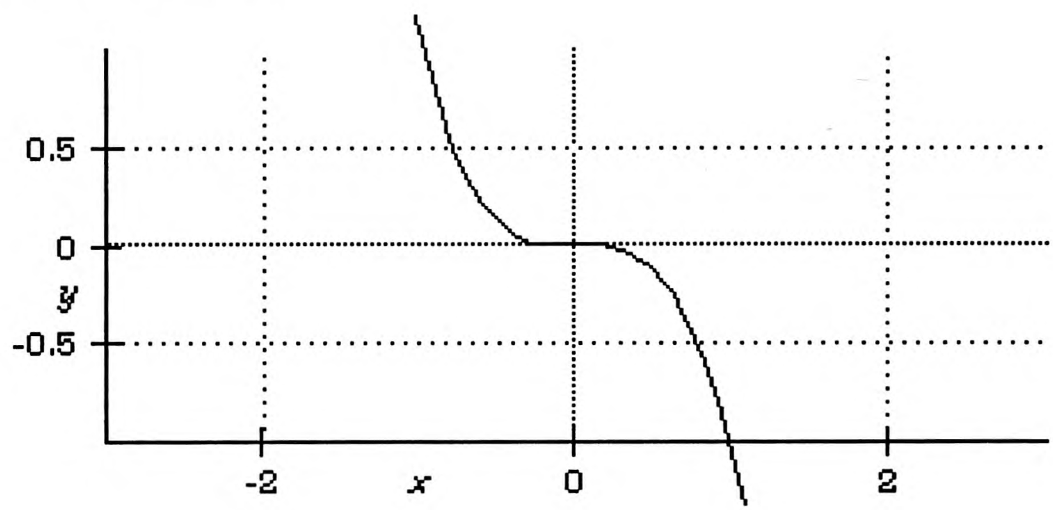
Now type **1** into the second-order coefficient, the x^2 one. Notice this is the first time you get curvature. It always looks like a parabola, either up or down, depending on the sign of the second coefficient. (Use the hand again to center the graph.)

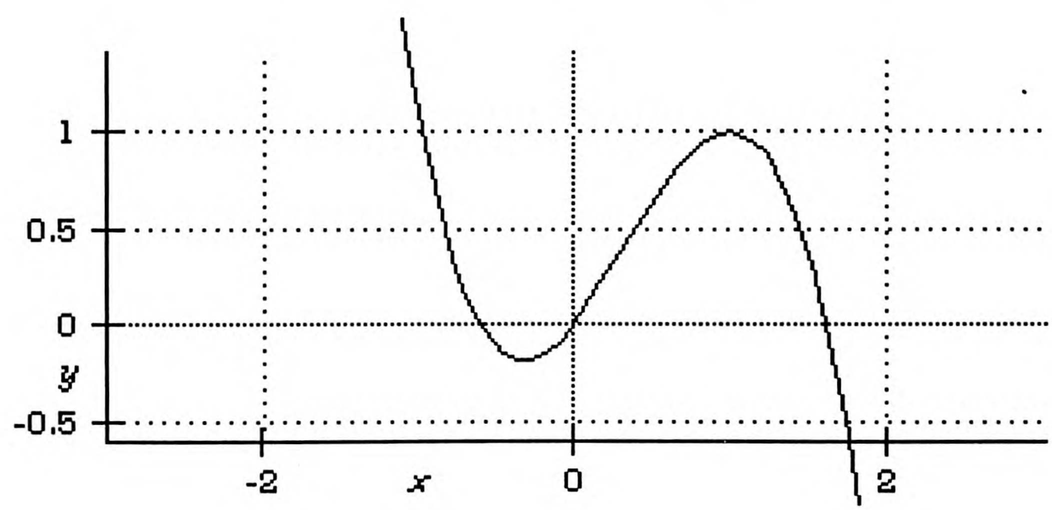


If you enter a 1 into the third-order coefficient, with zeros for the other coefficients, you get this:



If you then experiment with other combinations, you get graphs that look like the following:





Notice how the graph always has two humps, though they may be shallow. More terms and coefficients give the function more flexibility.

As you can see, the higher the order of the polynomial, the more degrees of freedom it has and the more different kinds of functions it can approximate. Later, in the Taylor Series tutorial, you learn how to generate a polynomial specifically designed to approximate a function.

MANIPULATING POLYNOMIALS

This tutorial provides examples of several manipulations used to manage polynomials and other expressions. The tutorial describes:

- Using the Expand and Factor manipulations
- Using the Collect manipulation to group particular parts of an expression together
- Using the Commute manipulation to reorder the terms of a polynomial
- Working with polynomials that contain two or more different variables
- Working with rational expressions (one polynomial divided by another)

Polynomials are found virtually everywhere in mathematics. They are so useful that some of the simpler symbolic algebra programs can represent nothing but polynomials.

Polynomials form a world unto themselves. You can multiply two polynomials to get another polynomial. You can raise a polynomial to an integer power and get another polynomial.

Expand

Type **y=(x+3)^6**.

Select the $(x+3)^6$ by clicking just below the 6:

$$y = (x+3)^6$$

Click (or choose **Expand** from the **Manipulate** menu) to expand the expression out. Theorist takes a moment and indicates it is busy with its rotating cube (Macintosh) or hourglass (Windows).

$$y = x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x + 729$$

Factor

Factor is the opposite of Expand. Select the expression after the equal sign (if it is not already selected) by double-clicking on any "+" sign. (Of course you can just drag over the whole expression.) Then click (or choose **Factor** from the **Manipulate** menu).

$$\triangle\ y = (x+3)^6$$

Factoring takes more time than Expanding because it is a difficult procedure akin to unscrambling an egg.

Collect

Collect is a stripped-down version of Factor. Type **y=12*(x^2)-6*(x^6)+3*x^5.**

Select the right side of the equation. (Remember the shortcut: double-click on the "+" sign.) Click (or choose **Collect** from the **Manipulate** menu).

$$\triangle\ y = 3\left(-2x^4 + x^3 + 4\right)x^2$$

Notice that Collect picks out $3x^2$ as a common factor to all the terms and it multiplies it on the outside. Also notice that it organized the inside in polynomial order with the highest power to the left.

Commute

Many people want their polynomials arranged in reverse order from the default. You can do this by selecting the polynomial $(-2x^4 + x^3 + 4)$ and choosing from the palette's sub-palette (or choosing **Commute** from the **Manipulate** menu).

$$\triangle\ y = 3\left(4 + x^3 - 2x^4\right)x^2$$

Two Variables

You can use Collect on even more complicated expressions. Type **z=(x^3)*(y^2)+2*x*y+((3*x)/y)+((y^2)/x^2).**

Choose **Clarify** from the **Notebook** menu to remove the unnecessary parentheses. On the screen it appears as:

$$\square\ z = x^3y^2 + 2xy + \frac{3x}{y} + \frac{y^2}{x^2}$$

Select the whole expression to the right of the equal sign (drag or double click a "+") and perform a Collect command.

$$\triangle\ z = \frac{3x^3 + \left(x^5+1\right)y^3 + 2x^3y^2}{x^2y}$$

Notice that it collects out a common denominator of x^2y. Anything in the denominator of any term is collected out as a common denominator.

Also note that it arranges the numerator as a polynomial in y, even collecting the x^5 and 1 together as a sort of coefficient for y^3. This is because y is chosen as the polynomial variable. You can control this. Theorist takes the last variable in the first term as the polynomial variable:

last variable

$$\underbrace{x^3y^2}_{\text{first term}} + 2xy +$$

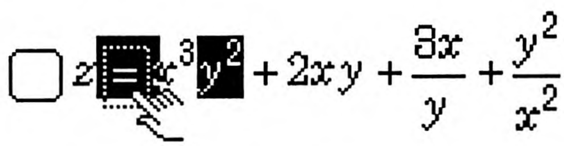
first term

Let's have Theorist consider x to be the polynomial variable in the assumption. Start with this again:

$$\square\; z = x^3y^2 + 2xy + \frac{3x}{y} + \frac{y^2}{x^2}$$

This is a more direct way of Commuting.

Select y^2 as shown below and hold the ⌘ key (Macintosh) or Ctrl key (Windows) as you drag the selection over to the left of the x^3. (The slot at the equal sign highlights.)



Drop in the y^2 by releasing the mouse button.

$$\triangle\; z = y^2x^3 + 2xy + \frac{3x}{y} + \frac{y^2}{x^2}$$

Select the expression to the right of the equal sign by dragging or double-clicking a "+", then Collect it.

$$\triangle\; z = \frac{\left(2y^2+3\right)x^3 + x^5y^3 + y^3}{x^2y}$$

This time it collects terms assuming x is the polynomial variable.

More Than Two Variables

Almost any expression can qualify as a polynomial, but if you have too many variables, Theorist can get confused. Fortunately, you can do things more carefully by hand. Make a new assumption and type **(x+y)*(z+a)**.

Select the whole expression and click (or choose **Expand** from the **Manipulate** menu).

$$\triangle (x+y)(z+\alpha) = \alpha x + \alpha y + x z + y z$$

If you try to factor this back with Factor, you get a confusing answer. Instead, do this three-step process:

❶ Select two terms that have a common factor, like this:

$$\triangle (x+y)(z+\alpha) = \alpha x + \alpha y + x z + y z$$

...then do a Collect.

$$\triangle (x+y)(z+\alpha) = \alpha x + \alpha y + (x+y)z$$

❷ Select the other two terms:

$$\triangle (x+y)(z+\alpha) = \alpha x + \alpha y + (x+y)z$$

...and do another Collect:

$$\triangle (x+y)(z+\alpha) = \alpha (x+y) + (x+y)z$$

❸ Select the expression to the right of the equal sign:

$$\triangle (x+y)(z+\alpha) = \alpha (x+y) + (x+y)z$$

...and do another Collect:

$$\triangle (x+y)(z+\alpha) = (x+y)(z+\alpha)$$

Rational Expressions

Rational expressions are also known as rational functions and sometimes referred to as rational fractions.

A rational expression is one polynomial divided by another. Create a new notebook and type **((x^2)+5*x+6)/x+3**. Select the whole expression and perform an Expand.

$$\triangle \frac{x^2+5x+6}{x+3} = x+2$$

When you Expand a rational expression, Theorist tries to divide the polynomials using long division. In this case, it divided evenly. Change the 5 in the assumption to 4. Theorist automatically redoes the Expand.

$$\frac{x^2+4x+6}{x+3} = 3\frac{1}{x+3} + x + 1$$

This time it does not divide out evenly and there is a remainder.

Randomly chosen rational expressions usually do not divide out evenly; you will have a remainder. The order of the numerator of the remainder is always less than the order of the denominator. For instance, you never have a remainder that looks like this,

$$\frac{x^2+x+4}{x^2-x+3}$$

...because this can be divided one more step, which Theorist does for you.

You can reverse the long division and get back to the original rational expression by simply selecting the whole expression to the right of the equal sign and doing a Collect:

$$\frac{x^2+4x+6}{x+3} = \frac{(x+3)x+x+6}{x+3}$$

Two Expands gives you what you started with. There are times when you need to break down a rational expression even further. For Integration and Laplace transforms you usually need to have denominators that are binomials. Here is how to do it...

In a new notebook, type **((x^3)+(2*x^2)-3*x-7)/(x^2)-4** and choose **Clarify** from the **Notebook** menu, which gives you on the screen:

$$\frac{x^3+2x^2-3x-7}{x^2-4}$$

The first step is to divide it out as before by selecting the entire expression and Expanding it:

$$\frac{x^3+2x^2-3x-7}{x^2-4} = \frac{x+1}{x^2-4} + x + 2$$

Now select the denominator of the right fraction, $x^2 - 4$, and click  (or choose **Factor** from the **Manipulate** menu).

$$\triangle\ \frac{x^3+2x^2-3x-7}{x^2-4} = \frac{x+1}{(x+2)(x-2)} + x + 2$$

Now select the whole right fraction and Expand. This time Theorist recognizes what you are trying to do and, instead of trying to divide the polynomials, does what is called a "partial fraction decomposition."

$$\triangle\ \frac{x^3+2x^2-3x-7}{x^2-4} = \left(\frac{1}{4}\frac{1}{x+2} + \frac{3}{4}\frac{1}{x-2}\right) + x + 2$$

This is the simplest form into which Theorist can put your rational expression.

QUADRATIC EQUATION

This tutorial works through the classic derivation of a solution to the quadratic equation ($ax^2 + bx + c = 0$). You could use the Factor manipulation to reach the solution directly, but this tutorial describes how to reach the same solution by hand.

The well-known formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is the solution of the quadratic equation $ax^2 + bx + c = 0$. Given a, b, and c, we will solve for x.

Make sure your new notebook's manipulation preferences are set so that Auto Simplify and Auto Casing are both on. That is, both **Auto Simplify** and **Auto Casing** must be checked under the **Manipulate▸ Preferences** submenu.

Enter the equation $ax^2 + bx + c = 0$, using your favorite method of equation entry.

First select $ax^2 + bx + c$ by double-clicking an addition sign. Click (or choose **Factor** from the **Manipulate** menu), which gives the answer:

$$a\left(x+\frac{1}{2}\sqrt{\frac{b^2}{a^2}-4\frac{c}{a}}+\frac{1}{2}\frac{b}{a}\right)\left(x-\frac{1}{2}\sqrt{\frac{b^2}{a^2}-4\frac{c}{a}}+\frac{1}{2}\frac{b}{a}\right)=0$$

But that would be too easy. Delete the answer and we will derive it the hard way...

Select the c and choose from the pop-up sub-palette (or choose **Move Over** from the **Manipulate** menu's **Other** submenu).

$$ax^2 + bx = -c$$

Next, we will divide both sides of the equation by a. Make sure the whole equation is selected and click (or choose **Apply** from the **Manipulate** menu), which selects both sides of the equation independently. Choose $\frac{a}{b}$ from the a+b pop-up to make fractions, then click on a from the variables palette or type **a** from the keyboard.

$$\triangle \; \frac{ax^2+bx}{a} = \frac{-c}{a}$$

Click on the equal sign in the equation (don't be alarmed by the beep, which signals the end of the Apply) and click (or choose **Expand** from the **Manipulate** menu).

$$\triangle \; \frac{bx}{a} + x^2 = -\frac{c}{a}$$

We want to make the expression on the left of the equal sign into a perfect square. The missing term is the square of half of b/a, which is $\left(\frac{b}{2a}\right)^2$.

Typing hint: **(b/(2*a))^2**.

Let's add this term to both sides of the equation. With the equation selected, click (or choose **Apply** from the **Manipulate** menu). Choose a+b from the a+b pop-up sub-palette, or type **+**. Enter the term $\left(\frac{b}{2a}\right)^2$ and you get:

$$\triangle \; \left(\frac{bx}{a} + x^2\right) + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Next, select everything on the left side of the equal sign and perform an Expand.

$$\triangle \; \frac{bx}{a} + x^2 + \frac{1}{4}\frac{b^2}{a^2} = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Now, create a new assumption. Enter the following, using your favorite method of equation entry.

$$\square \; \left(x + \frac{b}{2a}\right)^2$$

Select it and Expand.

$$\triangle \left(x+\frac{b}{2a}\right)^2 = \frac{bx}{a}+x^2+\frac{1}{4}\frac{b^2}{a^2}$$

Click on the equal sign to select the equation. Hold the ⌘ key (Macintosh) or Ctrl key (Windows) as you drag it over the left-hand side of the previous conclusion, like so:

$$\triangle \frac{bx}{a}+x^2+\frac{1}{4}\frac{b^2}{a^2} = -\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

Then release the mouse button to Substitute it.

$$\triangle \left(x+\frac{1}{2}\frac{b}{a}\right)^2 = -\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

At this point, make sure Auto Casing is on.

Select x and click (or choose **Isolate** from the **Manipulate** menu).

$$\triangle x = \sqrt{-\frac{c}{a}+\left(\frac{1}{2}\frac{b}{a}\right)^2}-\frac{1}{2}\frac{b}{a} \qquad \triangle x = -\sqrt{-\frac{c}{a}+\left(\frac{1}{2}\frac{b}{a}\right)^2}-\frac{1}{2}\frac{b}{a}$$

As you can see, Theorist saw there were both positive and negative square roots, so it put them in separate theories. Theorist only does this when its Auto Casing option is on.

Although the answers are in acceptable forms, we want to put them into the form we are familiar with. Let's start working with the left case theory. Select everything inside the radical sign then perform an Expand.

$$\triangle x = \sqrt{\frac{1}{4}\frac{b^2}{a^2}-\frac{c}{a}}-\frac{1}{2}\frac{b}{a}$$

With the inside of the radical selected, click (or choose **Collect** from the **Manipulate** menu).

$$\triangle x = \sqrt{\frac{1}{4}\frac{b^2-4ac}{a^2}}-\frac{1}{2}\frac{b}{a}$$

Select the entire expression to the right of the equal sign and do an Expand.

$$\triangle \; x = \frac{1}{2}\frac{\sqrt{b^2-4ac}}{a} - \frac{1}{2}\frac{b}{a}$$

Then, with the right side selected, perform a Collect.

$$\triangle \; x = \frac{1}{2}\frac{\sqrt{b^2-4ac}-b}{a}$$

Select the whole numerator and choose from the pop-up sub-palette (or choose **Commute** from the **Manipulate** menu's **Other** submenu), which gives us our final answer.

$$\triangle \; x = \frac{1}{2}\frac{-b+\sqrt{b^2-4ac}}{a}$$

ROOTS OF POLYNOMIALS

This tutorial describes how to find the roots of polynomials and describes the difference between expressions (which have zeros) and equations (which have roots). The tutorial also describes:

- How to create a graph of a polynomial to see its roots
- Using the Factor manipulation
- Using the UnCalculate manipulation to dispense with round-off errors

An expression is a group of symbols that evaluate to a number, matrix, or operator. An equation consists of two expressions connected by an equal sign. (Remember this distinction.)

One common task in mathematics is to find the value (or values) of x that makes a given equation true. These are the 'roots' of the equation.

From now on in this tutorial, use your favorite method of equation entry and editing.

Enter the equation $x^2 + 1 = 2x$ using your favorite entry method.

Choose **Clarify** from the **Notebook** menu to clean up any unnecessary parentheses. This gives you on your screen:

$$\square\; x^2 + 1 = 2x$$

Select the $2x$ and choose from the pop-up (or choose **Move Over** from the **Manipulate** menu's **Other** submenu). Or, you can simply hold the ⌘ key (Macintosh) or Ctrl key (Windows) as you drag it to the other side of the equal sign.

$$\triangle\; x^2 - 2x + 1 = 0 \quad \textit{MoveOver}$$

Remember: Expressions have zeros, equations have roots.

You can see that the roots of $x^2 + 1 = 2x$ are the same values that make the expression $x^2 - 2x + 1$ go to zero. These values are called the 'zeros' of the expression.

You can multiply polynomials together to get other polynomials. Make a new proposition and enter the expression:

$$\square\; (x-1)(x-1)$$

Select the whole expression and click (or choose **Expand** from the **Manipulate** menu).

$$\triangle\ (x-1)(x-1) = x^2 - 2x + 1 \quad \textit{Expand}$$

Whenever you multiply two binomials, you get a quadratic. In fact, whenever you multiply an M-order polynomial by an N-order polynomial, you get an (M + N)-order polynomial.

Exercise

Find out what you end up with when you raise an M-order polynomial to the Nth power. Try Expanding $(x^3 + 1)^5$.

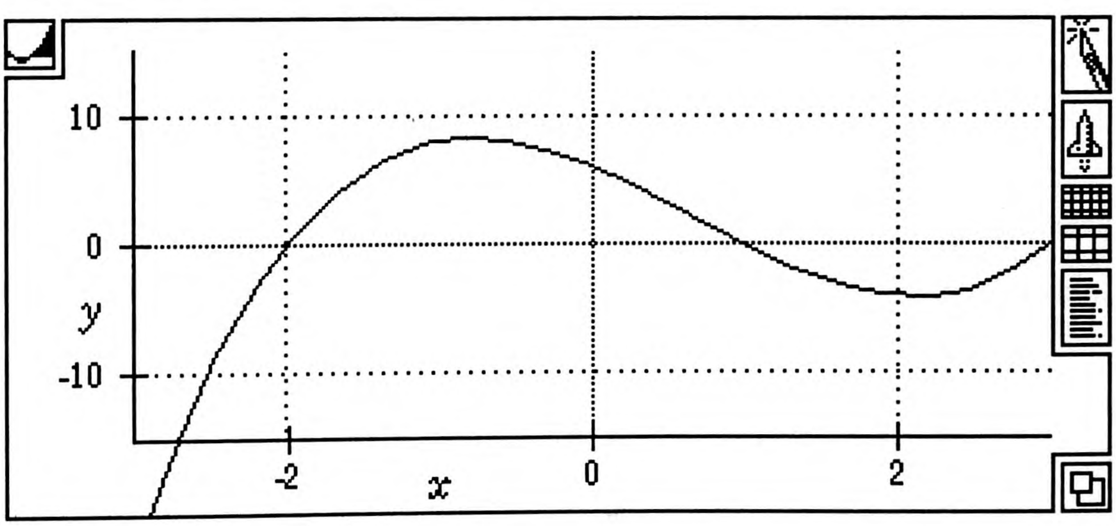
One way to think of polynomials is to factor them into binomials which, when multiplied together and expanded out, equal your polynomial. For example, $x^2 - 2x + 1$ could be thought of as $(x - 1)(x - 1)$ or $(x - 1)^2$.

An interesting side effect of this scheme is that the factors (the binomials from above) tell you of the zeros of the polynomial. For example, create a new proposition and then enter the equation:

$$\square\ y = (x-1)(x+2)(x-3)$$

Take a look at the equation. If $x = 1$, y goes to zero. The same is true if $x = -2$ or if $x = 3$. Everywhere else y is *not* zero. Select the expression to the right of the equal sign and Expand.

Graph the equation (by clicking or choosing **Graph▸y = f(x)▸ Linear**) to show where the roots are. (You may have to do some maneuvering with the Knife, Rocketship, and hand.) Notice the three zero crossings.



The Fundamental Theorem of Algebra says an N-order polynomial has N roots. Some roots might be complex. Some might be repeated. In fact, if all the coefficients are real, then the complex roots will come in complex conjugate pairs. For instance, the factors of $x^3 - x^2 + x - 1$ are $(x+i)(x-i)(x-1)$.

The task of finding roots and factors of polynomials has been plaguing mankind since the time of the Greek mathematicians. In fact it is not always possible to find a root in the form that you want, especially for large polynomials, or those with multiple variables. We will explore a variety of methods of finding roots. Fortunately, Theorist has many such tools available.

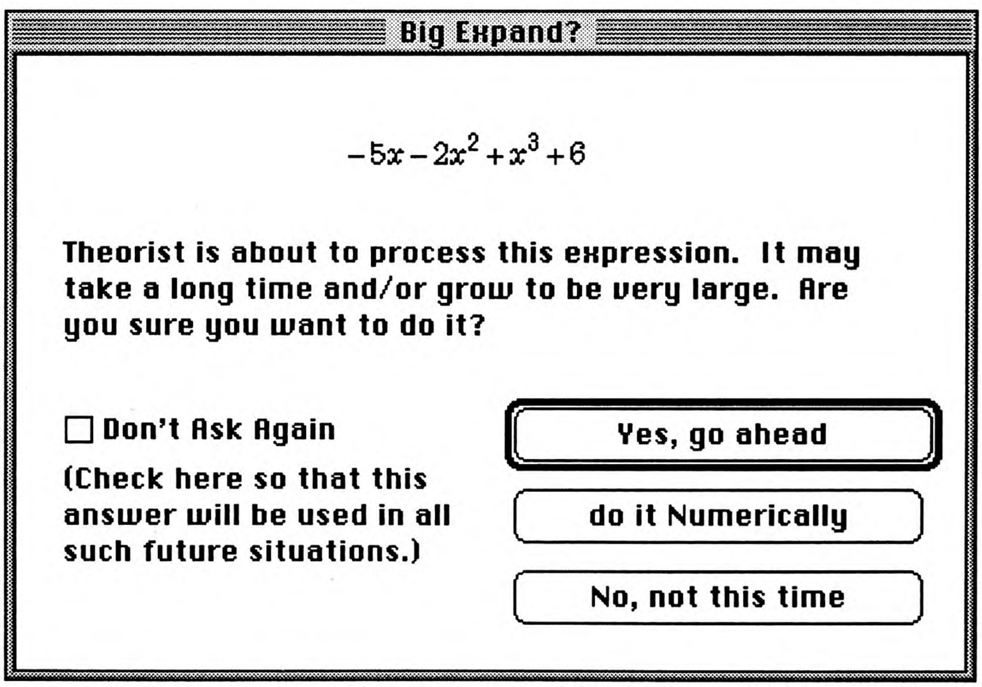
Factor Manipulation

In many cases, the fastest way to factor a polynomial is to use the Factor manipulation. In the equation:

$$y = x^3 - 2x^2 - 5x + 6$$

...select the expression to the right of the equal sign (remember that you can just double-click on the "+"). Click (or choose **Factor** from the **Manipulate** menu).

At this point you will likely encounter the following dialog:



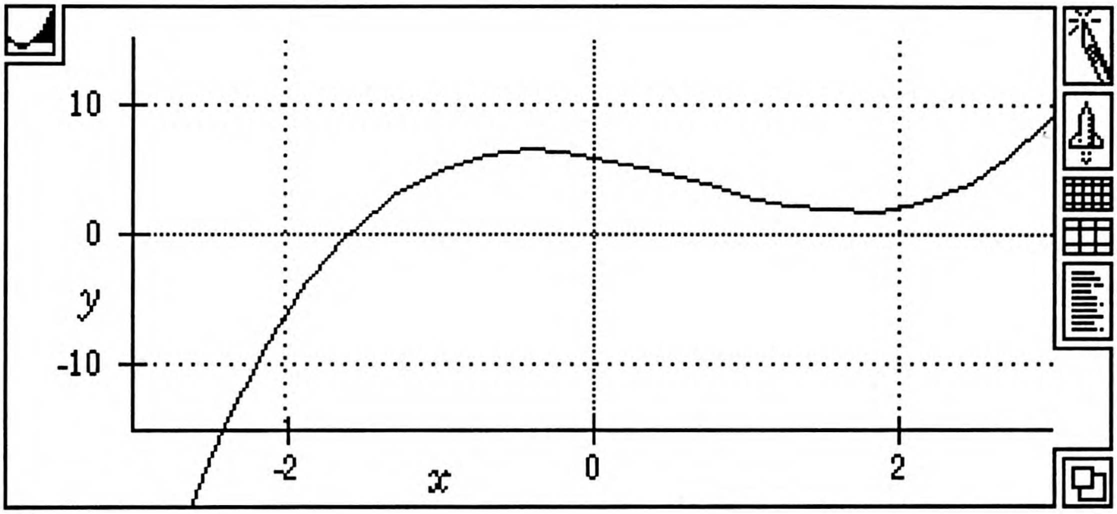
You can just accept the default to continue. (See the Theorist Reference Manual for details about this dialog.) Theorist works at it and then gives your answer.

$$y = (x-3)(x+2)(x-1) \quad \textit{Factor}$$

In some cases, your answer does not come out as cleanly. Change the 5 in the first conclusion to 2, which makes a new assumption:

$$\square\; y = x^3 - 2x^2 - 2x + 6$$

Choose **Make Working Stmt** from the **Notebook** menu. The old working statement for y automatically turns off. Have a look at the graph.



Notice that the graph no longer crosses zero in three places, just one. Changing the 5 to 2 raises the valley so it no longer dips below zero. The two roots look like they disappeared; in reality, they are on the complex plane. We will find them algebraically.

Now, in the working statement, select the expression to the right of the equal sign and Factor it. When the dialog appears, click the Yes button. It takes a while and produces a large answer.

$$\triangle\; y = \left(x + \frac{1}{3}\cdot 100^{\frac{1}{3}} + \frac{1}{3}\cdot 10^{\frac{1}{3}} - \frac{2}{3}\right)\cdots\cdots \qquad \textit{Factor}$$

By clicking Yes, we asked for the exact, analytical answer. If our polynomial had something other than numbers for coefficients, the result might be more useful.

Do this Factor over again, but when the dialog comes up, this time click on the button for a numerical answer, which gives us an answer we can use.

$$\triangle\; y = (x + 1.5987)(x - 1.7993 + 0.71798i)(x + \cdots\cdots)$$

These roots seem unrelated to the original ones; two of them are complex. Most roots of randomly chosen polynomials are like these. You can verify that these are, in fact, the correct roots. Simply do an Expand of the right side (which you probably have selected).

Theorist can factor some rather large polynomials. Select the "6" in your polynomial and type **+x^6** so you get:

$$y = x^3 - 2x^2 - 2x + 6 + x^6$$

Then, try to Factor the right side. (You have to do it; do not rely on Theorist to remanipulate it.)

$$y = (x - 0.16923 + 1.492i)(x - 0.16923 - 1.492i)\cdots\cdots$$

Note that in this case, Theorist does not ask you whether you want to factor it numerically; there is no choice. If your polynomial is cubic or quartic, Theorist uses the cubic formula or the quartic formula. Discovered during the Renaissance, these formulas always work, and the answer will always be exact, although it may be large and complex (like our answer with cube roots, above) and Theorist may take a long time to derive it.

If your polynomial has numerical coefficients, Theorist can find the roots using a numerical algorithm, even if the polynomial is larger than a quartic. The result, though, is not entirely exact, as we shall see. Expand this polynomial again to see if you get the original back, as you did before:

$$y = x^6 - 1.5282\times10^{-13}x^5 + x^3 - 2x^2 - 8.8818\times10^{-16}ix - 2x + 6 \quad \textit{Expand}$$

Note that it is not the same; there are several new terms. All of these are very small, in scientific notation. Many are imaginary. What happened was that the roots were computed to about the maximum 15 digits of precision available in Theorist. (The last few digits of a root are extremely hard to find). When you multiply them all back together, all of the coefficients that are supposed to be zero end up being close to zero, meaning they end up being ugly tiny numbers like you see here. If you set Theorist's display precision to 15 digits, you will see even the "good" numbers have some rough edges due to round-off error.

UnCalculate Manipulation

Theorist has a manipulation to patch this up. Keeping your current selection, choose from the palette's pop-up sub-palette (or choose **UnCalculate** from the **Manipulate** menu's **Other** submenu). You get your original polynomial.

$$y = x^6 + x^3 - 2x^2 - 2x + 6 \quad \textit{UnCalculate}$$

UnCalculate examines all of the numbers in the expression and tries to figure out what they "should" be. Numbers that are close to integers, or to certain rational numbers, are made exact integers or rationals. UnCalculate also recognizes some square roots and other forms. (See the Theorist Reference Manual for more details on UnCalculate.)

Note that UnCalculate will actually change the value of the expression slightly. You should be careful when using it. There are some situations where it is not appropriate to use it. Take for instance this expression expanded out:

$$\triangle\ (x - 1\times10^{-5})^5 = x^5 - 5\times10^{-5}x^4 + 1\times10^{-9}x^3 - \cdots\cdots$$

If you UnCalculate it, Theorist looks at the very small coefficients and assumes they are supposed to be zero, and produces this result:

$$\triangle\ (x - 1\times10^{-5})^5 = x^5 - 5\times10^{-5}x^4 + 1\times10^{-9}x^3$$

All of the extra terms are lost. If x is of the order of 10^{-5}, then this is not an appropriate thing to do, as these lost terms were significant and about half of the significant terms in the polynomial were lost. You must use good judgment when using the UnCalculate manipulation.

NONLINEAR SYSTEM

This tutorial describes how to work with nonlinear systems that include the product of independent variables. The trick is to treat these products, temporarily, as single variables. The tutorial provides several examples of the Substitute, Collect, Isolate, and Simplify manipulations.

In a new notebook, enter the following equations:

$$xy + yz - zx = a$$
$$xy - yz + zx = b$$
$$-xy + yz + zx = c$$

(Remember that *xy* means the variables *x* and *y* are multiplied together. If you fail to enter the multiplication symbol, you get non-italicized xy, which is a single name and will cause this tutorial to fail.) Open the notebook's Declarations and change the name declarations for *a, b,* and *c* from constants to variables. Make sure the **Auto Casing** option is *not* checked—that is, *off*—under the **Manipulate▸Preferences** submenu.

You need to solve for *x*, *y*, and *z*. The usual strategy is to take an equation, solve for a variable, then plug that substitution into the other equations. If you do that here you quickly end up with a mess.

With that in mind, modify your original strategy slightly: pretend that *xy, yz,* and *zx* are all variables, and solve for them first. This gives you another three equations in three unknowns; apply the strategy again to solve them and get the final answer.

Start by isolating *xy* from the first equation, then take that answer and Substitute it into the other two equations.

$$xy + yz - zx = a$$
$$xy = a + xz - yz \quad \text{Isolate}$$
$$xy - yz + zx = b$$
$$a + 2xz - 2yz = b \quad \text{Substitute}$$
$$-xy + yz + zx = c$$
$$-a + 2yz = c \quad \text{Substitute}$$

To make the notebook easier to follow, take the conclusions and arrange them as shown:

$$xy = a + xz - yz$$ *Isolate*

$$a + 2xz - 2yz = b$$ *Substitute*

$$-a + 2yz = c$$ *Substitute*

Now, with the last two equations, you have two equations in two unknowns. In fact you are lucky here because the last equation solves immediately for yz. Isolate yz from the $-a + 2yz = c$ equation now.

$$yz = \frac{1}{2}(a + c)$$ *Isolate*

Now take the yz equation and substitute it into the middle equation, which gives you:

$$-c + 2xz = b$$ *Substitute*

Get a solution for xz by isolating it:

$$xz = \frac{1}{2}(b + c)$$ *Isolate*

...and substitute the xz equation into the $xy = a + xz - yz$ equation:

$$xy = \frac{1}{2}(b + c) + a - yz$$ *Substitute*

Into that, substitute the yz equation and Expand the result.

$$xy = \frac{1}{2}a + \frac{1}{2}b$$

The difficult part is done. To make it pretty, Collect everything to the right of the equal sign.

$$xy = \frac{1}{2}(a + b)$$ *Collect*

Here then are the (rearranged) three equations in three unknowns:

$$\triangle\; xy = \frac{1}{2}(a + b)$$

$$\triangle\; yz = \frac{1}{2}(a + c)$$

$$\triangle\; xz = \frac{1}{2}(b + c)$$

(You can drag them around to position them as above for the next steps.)

Start solving for *x, y,* and *z*. Isolate *x* from the *xy* equation.

$$\triangle\; x = \frac{1}{2}\frac{a + b}{y} \qquad \textit{Isolate}$$

Substitute it into the *xz* equation.

$$\triangle\; \frac{1}{2}\frac{(a + b)z}{y} = \frac{1}{2}(b + c) \qquad \textit{Substitute}$$

Next, Isolate *y* from the *yz* equation,

$$\triangle\; y = \frac{1}{2}\frac{a + c}{z} \qquad \textit{Isolate}$$

...and Substitute it into the previous equation.

$$\triangle\; \frac{(a + b)z^2}{a + c} = \frac{1}{2}(b + c) \qquad \textit{Substitute}$$

Now, solve for *z* by isolating it from the previous equation.

$$\triangle\; z = \sqrt{\frac{1}{2}\frac{(a + c)(b + c)}{a + b}} \qquad \textit{Isolate}$$

Substitute the *z* equation into the equation for *y*.

$$\triangle\; y = \frac{1}{2}(a + c)\frac{1}{\sqrt{\frac{1}{2}\frac{(a + c)(b + c)}{a + b}}} \qquad \textit{Substitute}$$

Expand the square root, then Simplify the whole thing.

$$\triangle \; y = \frac{1}{2}\frac{\sqrt{2}\sqrt{a+b}\sqrt{a+c}}{\sqrt{b+c}} \qquad \textit{Simplify}$$

Take that and substitute it into the equation for x.

$$\triangle \; x = \frac{1}{2}\frac{\sqrt{2}\sqrt{a+b}\sqrt{b+c}}{\sqrt{a+c}} \qquad \textit{Substitute}$$

If you Expand the square root in the z equation, you can see the symmetry.

$$\triangle \; x = \frac{1}{2}\frac{\sqrt{2}\sqrt{a+b}\sqrt{b+c}}{\sqrt{a+c}}$$

$$\triangle \; y = \frac{1}{2}\frac{\sqrt{2}\sqrt{a+b}\sqrt{a+c}}{\sqrt{b+c}}$$

$$\triangle \; z = \frac{1}{2}\frac{\sqrt{2}\sqrt{a+c}\sqrt{b+c}}{\sqrt{a+b}}$$

Solving Nonlinear Equations

In this tutorial, we will be using some of the rules and trigonometric identities defined in the New Notebook to work with more complicated nonlinear systems. The tutorial also describes:

- Using the Auto Casing option to see all solutions to a problem
- Using the Apply manipulation to change the appearance of an expression
- Using the Transform manipulation to execute a rule
- Using the Find Root command to calculate an apparent root shown in a graph
- Working with multiple nonlinear equations
- Creating Zero Contour plots as a technique for solving multiple nonlinear equations

Previous tutorials showed you how to solve one or more equations of the following forms:

❶ an equation that has one occurrence of x (just isolate x)
❷ an equation of the form polynomial = 0 (factor the polynomial; gives multiple answers)
❸ a set of linear equations (isolate and substitute, or make a matrix equation)
❹ a set of nonlinear equations with a pattern

This chapter is concerned with "other" equations or sets of equations. In general, this problem is more difficult because there are so many kinds of problems that need to be solved. Let's start with one equation.

(Throughout this discussion, we will assume that you are trying to solve for the variable x, or for the variables $x, y, z, a, b, \ldots$)

One Equation

Enter the following:

□ $x^2 = 3389$

Isolate x to get:

△ $x = \sqrt{3389}$ *Isolate*

Calculate the square root to get:

$$\triangle\ x = 58.215 \quad \textit{Calculate}$$

It is usually best to get an algebraic solution. Not only do you get an exact answer, but you get an answer from which you can gain more insight.

Delete the two conclusions (to avoid tainting the next thing we are going to do). Turn on Auto Casing and Isolate again to get:

$$\bigcirc\ \triangle\ x = \sqrt{3389} \qquad \bigcirc\ \triangle\ x = -\sqrt{3389}$$

Many equations have more than one solution. Theorist can frequently give you all of the solutions, if it is a simple algebraic situation like this.

The reason you had to delete the old answer is Theorist feels around for some insight about the variable x before it goes ahead and makes multiple cases. If it finds a working definition like "x = positive number" or "x > non-negative number", then Theorist will assume x is positive and the negative case is not applicable, so it will only generate one case, even if Auto Casing is on.

Delete the assumption and case theories. Enter:

$$\square\ 0.66744 = e^{0.0174}\sin(33.2x)$$

Auto Casing should be left on.

Isolate x. You get two cases, one for the uphill side of the sine, and one for the downhill side (which is negated and offset by π).

$$\bigcirc\ \triangle\ x = 0.03012\left(2\pi n_{107} + \arcsin\left[0.66744\frac{1}{e^{0.0174}}\right]\right)$$

$$\bigcirc\ \triangle\ x = 0.03012\left(2\pi n_{106} - \arcsin\left[0.66744\frac{1}{e^{0.0174}}\right] + \pi\right)$$

The two constants n_{106} and n_{107} are arbitrary integers. In other words, each case represents a countable infinity of solutions. (An arbitrary integer represents a countable infinity, because you can count them. An arbitrary real number represents an uncountable infinity of solutions.)

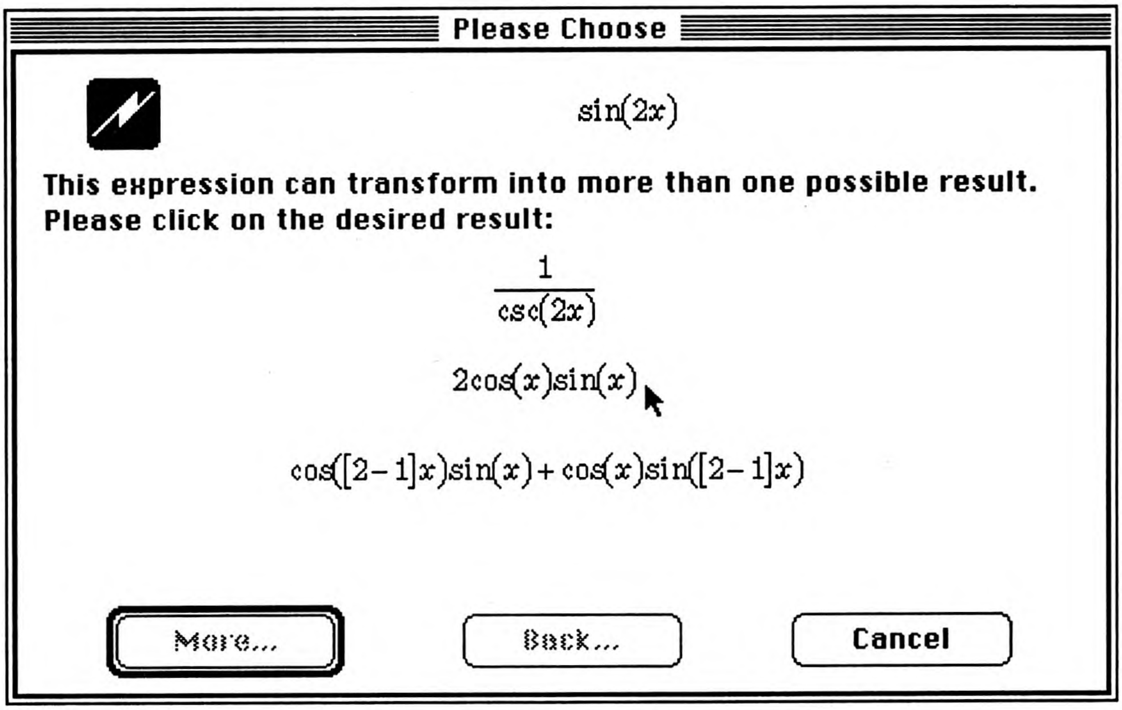
If your equation has more than one occurrence of x, try to get to one occurrence by using some algebra.

Theorist uses transformation rules previously set up in the New Notebook.

Delete the assumption and case theories. Enter the following.

$$\square\ \sin(2x) = \tan(x)$$

Select the equation and click (or choose **Apply** from the **Manipulate** menu). Click (or choose **Transform** from the **Manipulate** menu). Since the notebook has more than one rule which matches, a dialog asks you to choose a form for the result.



Click the 2 cos(x) sin(x) transformation result.

$$\triangle\ 2\cos(x)\sin(x) = \frac{\sin(x)}{\cos(x)}$$

Turn off Auto Casing so the next few steps will be easier. Move the sine on the left to the right, then move the cosine on the right to the left, so you get this:

$$\triangle\ 2(\cos[x])^2 = 1$$

Then, solve for x by isolating.

$$\triangle\ x = \frac{1}{4}\pi$$

If you turn on Auto Casing, you will get many more solutions. For instance, $x = 0$ is a solution, and the solutions repeat, as you would expect trig functions to do. (Many other symbolic algebra programs actually fail to give all of the answers for this equation.)

In the event that your equation is not so fortunate that you can move the x's together, you can use the numerical root finder in Theorist. This situation comes up a lot in real life; the problems that you are given in math books (and tutorials like this) tend to be "easy" to solve.

Enter:

$$\frac{1}{8}x+0.1=\sin(x)$$

Our first step is to convert the equation to the form "0 = everything else," so select the left-hand side of the equation and move it over to the right.

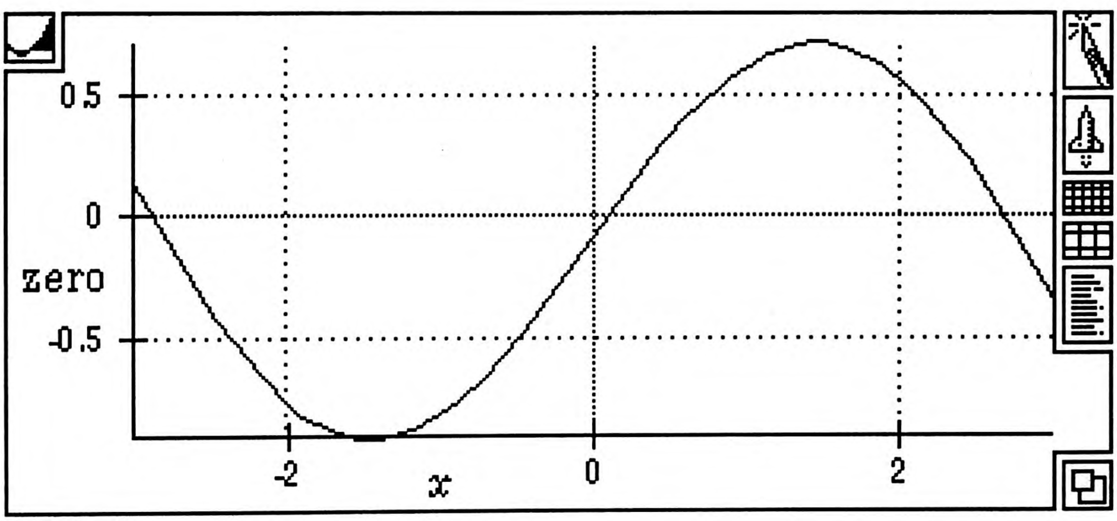
$$0=\sin(x)-\frac{1}{8}x-0.1$$

Next, change the "0" to "zero".

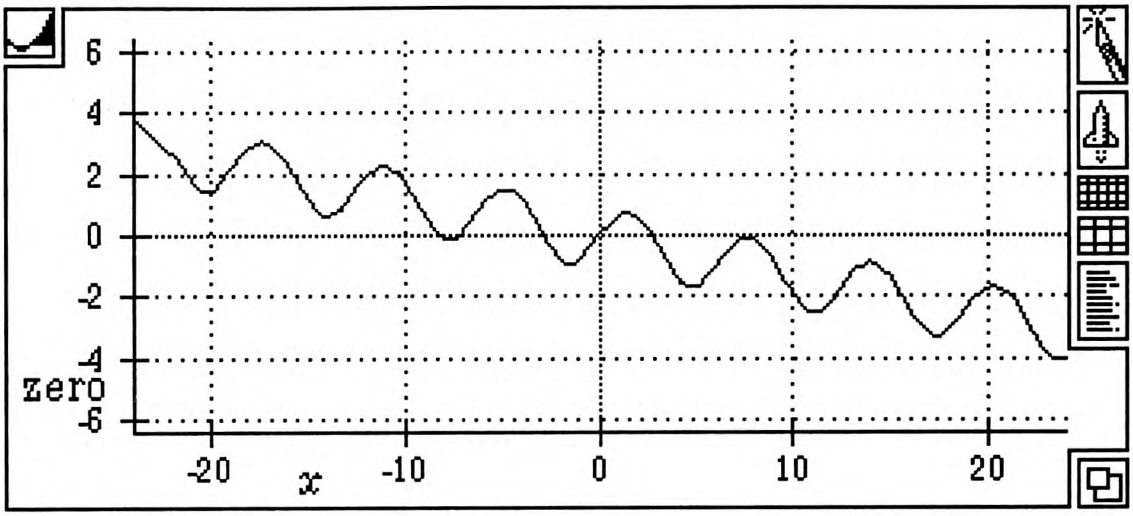
$$\text{zero}=\sin(x)-\frac{1}{8}x-0.1$$

When Theorist gets around to declaring "zero" call it a variable.

Now, click (or choose **Graph▸y = f(x)▸Linear**) to see what this looks like. Our job, now, is to make zero be equal to 0, so when we look at the graph, we are interested in the zero crossings.



Well, we can see three roots here. Let's zoom out to see if there are more. Click on the Rocketship icon three times and we can see that there's potential for a few more roots.

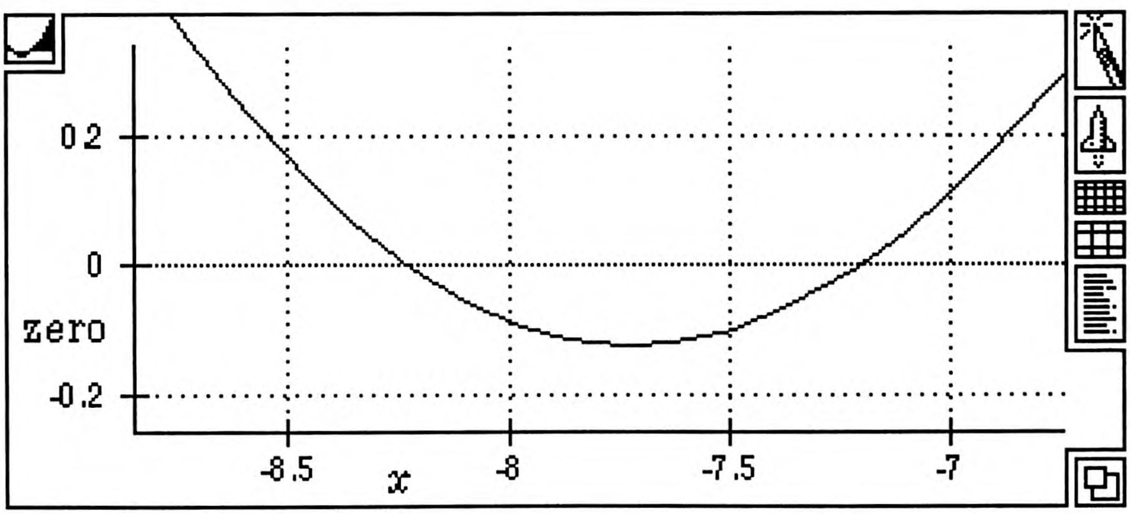


Besides the original three, the curve looks like it dips just below zero on the left, and it almost reaches above zero on the right. With the Knife and Rocketship you can explore these and verify for yourself that at –8 or so it crosses twice, and at 8 it never quite crosses.

In the book *Numerical Recipes* (Press, Flannery, Teukolsky and Vetterling, Cambridge University Press), the authors even go so far as to supply a routine to draw a crude asterisk-plot of the function you are working with, just so you can see what you are doing and avoid entrusting your Ph.D thesis to a simple-minded blind algorithm. Using Theorist is much easier.

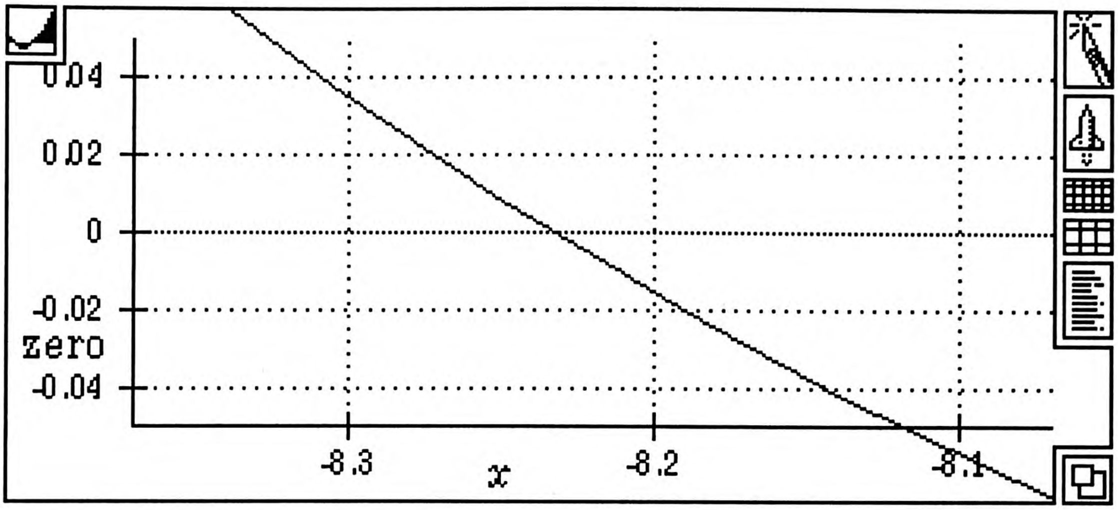
This curve demonstrates some classical pitfalls of the routine practice of having a blind computational algorithm finding one or more roots for you: there may be more than one; there may be "almost" roots (as in this case, near 8); the roots may be closely spaced and look like they are "almost" roots (as in this case, near –8), or there may be valleys that can lead your root-finding program on a wild goose chase.

We can find the left-most root by zooming in on it with the knife. Do so. From what we can see in this view, it looks to be about –8.25:



We could continue to slice with the knife so that we get more and more digits of precision. With practice, you can get about one more digit of precision with each slice. Not only is this tedious, but you will find that the graphing system breaks down at a certain point due to lack of precision.

A better approach is to use the Find Graph Root command. Once we have isolated the zero crossing with another knife stroke,



...we can choose **Find Graph Root** from the **Manipulate** menu, which calculates the root to the full 14 or 15 digits of accuracy:

$$x = -8.2328$$
$$\text{zero} = 8.3267\times10^{-17}$$

The first equation gives us a value of x. The second gives us a value of "zero" at that point, which should be close to zero. (Even though only five digits are shown here, all 15 digits in these numbers are available by changing the display precision from the Notebook menu. The root finding algorithm attempts to calculate an answer that is accurate to the full precision.)

Theorist puts the solution into a case theory because the situation where x is this value is not the whole of our work (for instance, we want to consider other roots at other values of x). You can continue any derivation you need to do with that root in its case theory.

By repeating this process, we can find all of the other roots to great precision.

When using this technique, you have the freedom to arrange the equation that you use to draw the graph a little differently. For instance, if you needed to solve $1/x = x$, the obvious thing is to turn it into $0 = -1/x + x$. By moving things around a bit first, you can arrive at $1 = x^2$ and then $0 = x^2 - 1$. Both equations have zeros at the same places, and perhaps the second one is easier to deal with because there is no singularity at $x = 0$.

Multiple Equations

Solving multiple equations may be more difficult than a single equation. Some of the other tutorials have shown some tricks you can use for certain types of equations:

❶ If the equations are all linear, make it a matrix equation.
❷ If the equations have some special symmetry, you can exploit that.

In the usual case, though, there are few tricks you can pull. The standard algorithm for solving N equations in N unknowns is an extension of what you saw in solving two equations in two unknowns:

❶ Algebraically eliminate one variable. Usually you end up solving one of the equations for a variable, "variable = other stuff", then plugging the other stuff in for the variable in all of the other equations. You now have $N - 1$ equations in $N - 1$ unknowns.
❷ Solve the $N - 1$ equation system. (This is a recursive algorithm.)
❸ Plug the $N - 1$ values for the other variables into your equation "variable = other stuff", and get a value for this variable.

Here's a simple demonstration of that procedure. In a new notebook, enter the following system.

$$\square\; xyz = -6$$
$$\square\; x = -y \cdot 2^z$$
$$\square\; x + y + z = -4$$

Make sure the Auto Casing preference is off.

First, let's eliminate one variable, x, and one equation, the first equation. Solve for x in the first equation.

$$\triangle\; x = -6\frac{1}{y\,z} \quad \textit{Isolate}$$

Then, Substitute that value of x into the other two equations. Each equation turns into a new equation, and the two of them together constitute the new system of two equations in two unknowns.

$$\triangle\; -6\frac{1}{y\,z} = -2^z\,y \quad \textit{Substitute}$$
$$\triangle\; -6\frac{1}{y\,z} + y + z = -4 \quad \textit{Substitute}$$

Drag the $\frac{1}{yz}$ from the left side over to the right side of the equation:

$$-6\frac{1}{y\,z} = -2^{z}\,y$$

Which unites the two y's into y^2.

$$-6 = -2^{z}\,y^{2}z \qquad \textit{MoveOver}$$

We then solve for y:

$$y = \sqrt{6\frac{2^{-z}}{z}} \qquad \textit{Isolate}$$

Substitute that answer for y into the other equation, like so:

$$-6\frac{1}{y\,z} + y + z = -4 \qquad \textit{Substitute}$$

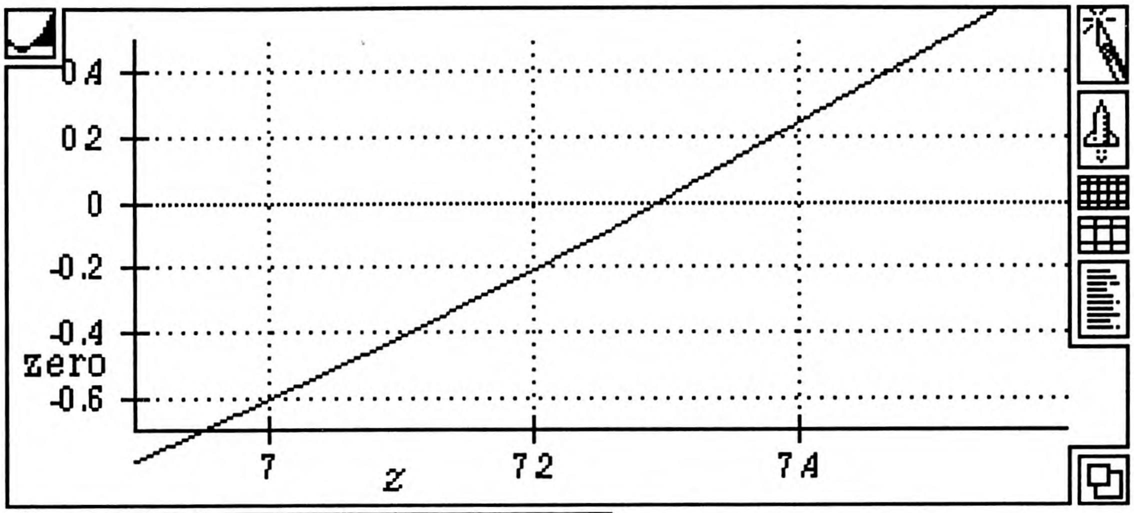
This results in an unsightly equation.

$$-6\frac{1}{z}\frac{1}{\sqrt{6\frac{2^{-z}}{z}}} + \sqrt{6\frac{2^{-z}}{z}} + z = -4 \qquad \textit{Substitute}$$

Take the left side and move it to the right side to put it in the proper form:

$$0 = 6\frac{1}{z}\frac{1}{\sqrt{6\frac{2^{-z}}{z}}} - \sqrt{6\frac{2^{-z}}{z}} - z - 4 \qquad \textit{MoveOver}$$

Type **zero** in place of the 0, and draw a graph. By using the Rocketship and knife and then using the Find Graph Root command, we can find a root near 7.3:



$z = 7.2961$

$\text{zero} = 1.7764\times10^{-15}$

We take this value of *z*, and substitute it into our equation for *y*.

$$y = \sqrt{0.82236\frac{1}{2^{7.2961}}} \qquad \textit{Substitute}$$

Calculate the right side to a number.

$$y = 0.072337 \qquad \textit{Calculate}$$

Now that we have values for *y* and *z*, we can substitute them into the equation for *x*, the first conclusion we derived.

$$x = -11.368 \qquad \textit{Substitute}$$

Exercise

Substitute these values into the three original equations and verify that they work.

Sometimes life is not so easy. In the last example, the whole process of finding the solution amounted to repeatedly solving equations for one variable in terms of other variables. Depending on the equations you have, this may be easy, difficult, or impossible. The algorithm is flexible in that it does not dictate which variable to eliminate with which equation; you have the freedom to choose the variable and equation that you eliminate at each step to make your work easier (or to make it possible, as the case may be).

Sometimes life still isn't so easy! Usually you get to a place where you have M equations in M unknowns and none of them can be algebraically solved for one of the variables because they are just too intertwined. If you are left with two equations in two unknowns, there is another solution: you can use the two dimensional root finder in Theorist.

Let's work with two equations from the previous derivation. Copy and paste each into a new notebook.

$$\square\ -6\frac{1}{y\ z} = -2^{x}\ y$$

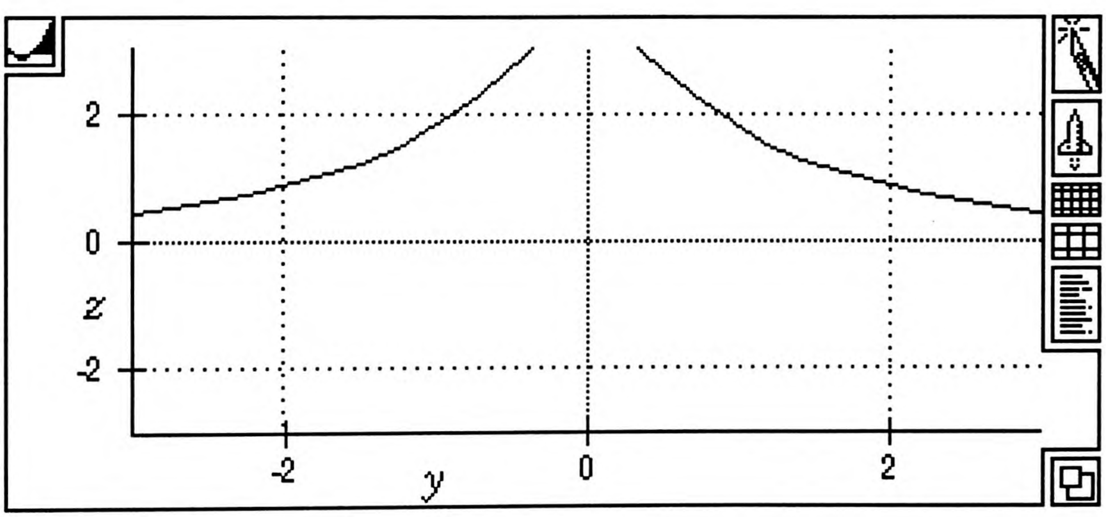
$$\square\ -6\frac{1}{y\ z} + y + z = -4$$

We'll start by getting a graph that shows both of the equations as zero-contour plots.

Take the first of these equations, and move the $1/_{yz}$ to the other side of the equation.

$$\triangle\ -6 = -2^{x}\ y^{2} z \qquad \textit{MoveOver}$$

The reason for this is infinities bother contour plots, so we try to eliminate reciprocals as much as possible. With the cursor still in this equation, choose **f(x,y) = g(x,y) Implicit** from the **Graph** menu's **Other** submenu. When the dialog comes up, change the axis variables to y and z. This makes a special contour graph showing where our expression goes to zero.



Note we are not looking for a zero crossing here. This line shows all of the (y, z) combinations where the first equation is satisfied. Of course, this is not enough; we want both equations satisfied at once.

Next, we will add in a zero contour for the other equation. Isolate the reciprocal on one side of the equation.

$$6\frac{1}{yz} = y + z + 4 \qquad \textit{Isolate}$$

Then, move the $1/yz$ to the other side, as you did with the first equation.

$$6 = (y + z + 4)yz \qquad \textit{MoveOver}$$

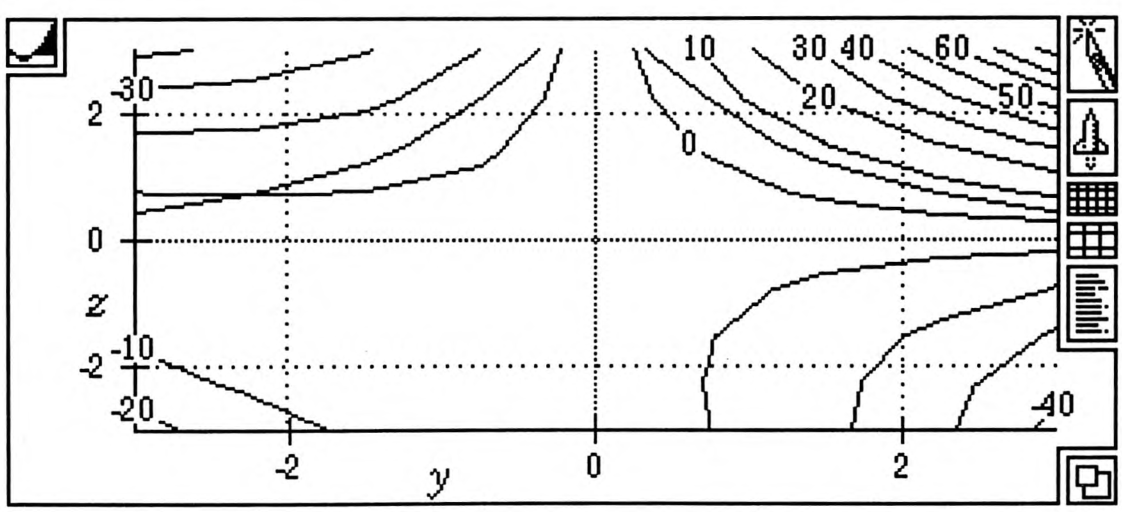
Move the 6 over.

$$0 = (y + z + 4)yz - 6 \qquad \textit{MoveOver}$$

Select 0 and type **zero**.

$$\text{zero} = (y + z + 4)yz - 6$$

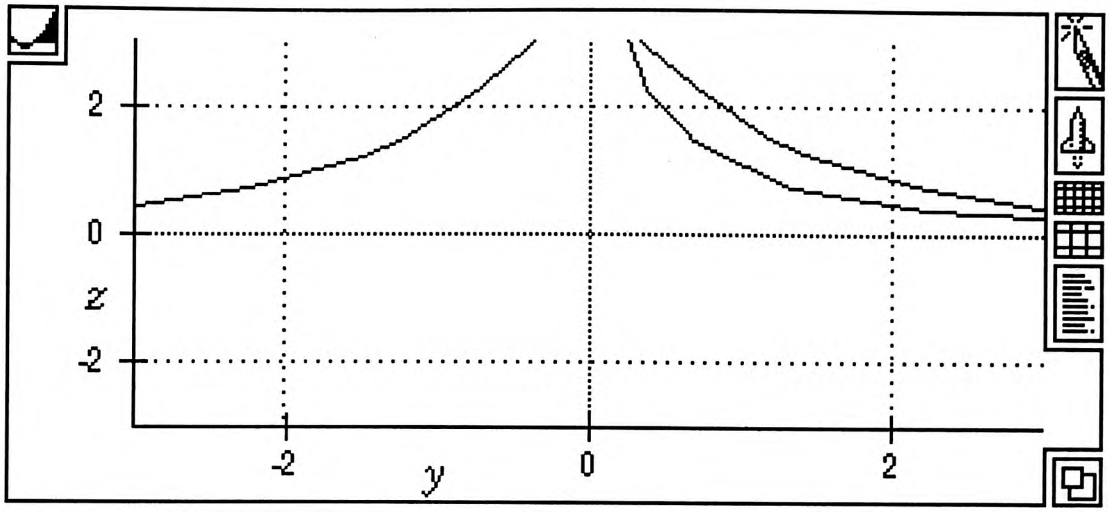
Clarify and declare zero as a variable. With the cursor still in this equation, choose **Add Contour Plot** from the **Graph** menu's **Additional** submenu. When the axis variables dialog comes up, make sure y and z are selected in the same order as before (this is to make sure that both contour plots are plotting y against y and z against z, instead of y against z) and use zero for the A axis. This makes the graph look like a mess.



Open the graph details and locate this added contour plot. (It will be at the bottom.) Change it so it says “at zero only” instead of “normally”.

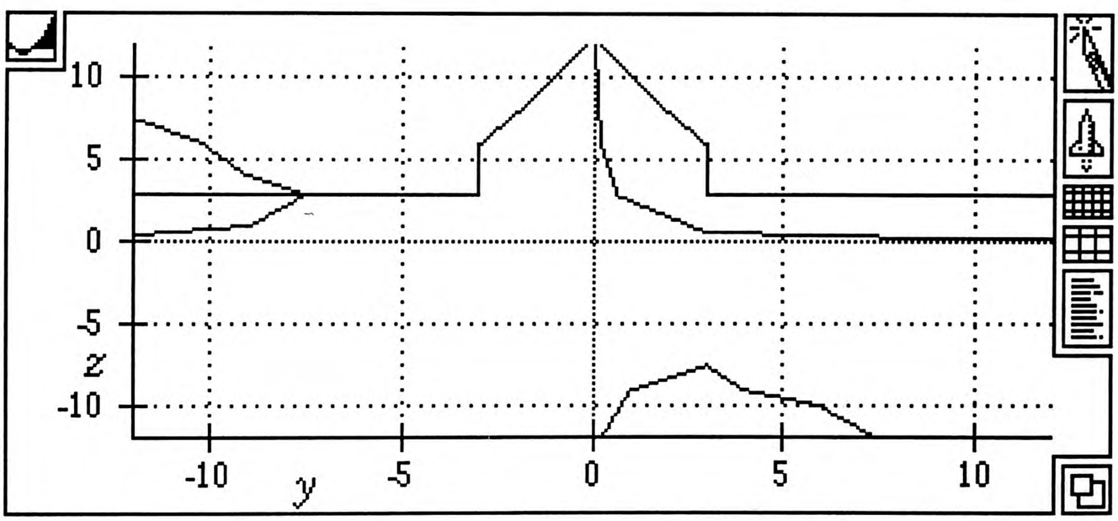
Contours at (y, z) where y = left … right and z = bottom … top; [at zero only ▾] spaced contours of zero.

The graph becomes more simple.



Now we have the graph the way we want it: each line shows where one or the other equation is satisfied. All we have to do is find intersection points. Two directions look promising: one is above and to the right; the other becomes apparent far to the left after zooming out two steps.

A contour plot only a mother could love. The reason it is so ugly is explained below. In fact, your graph may not even look like this. You can try increasing the graph’s accuracy to improve its looks.



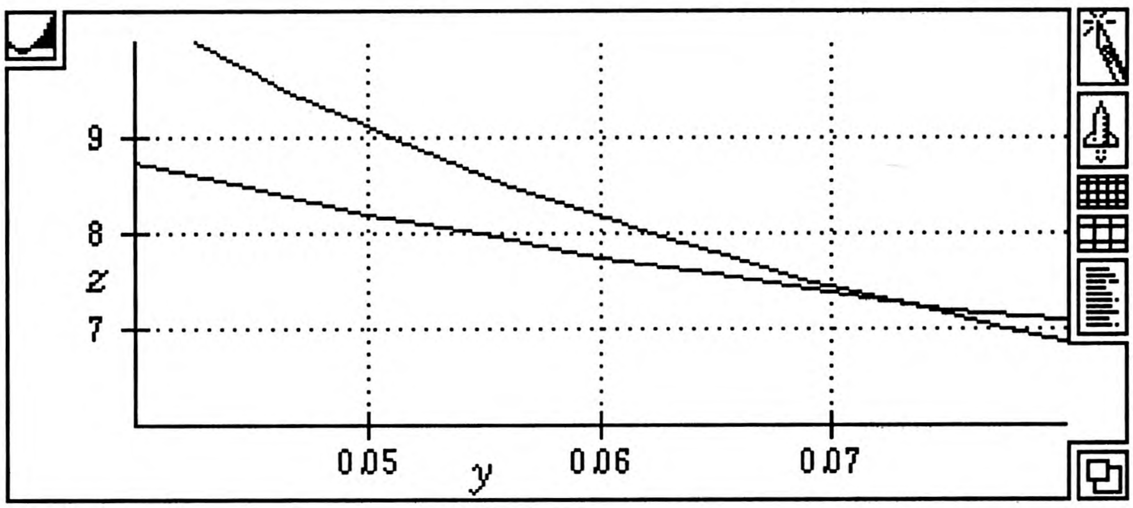
It also reveals some limitations of contour plots. The contour plot has to evaluate its function over the entire plane. This is slow. It partially compensates for this by sampling a sparse mesh of points and doing linear interpolation to find the zeros. Sometimes a linear approximation is poor, as you can see in this graph. There are two solutions to this problem:

❶ Increase accuracy. The default breaks up the graph area into 64 rectangles, evaluating the function at the 81 corners. Each click on the dense mesh button doubles the mesh density in each direction. A side effect is that this slows down the graphics by a factor of 4.

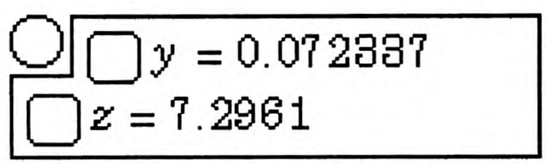
❷ Zoom in, carefully, to the points of interest to see what is really going on there. Do not zoom in too fast; you can't see where to zoom into very accurately. Two lines that pass close to each other may intersect if you zoom in closer.

Let's start by exploring the upper intersection. With the knife, zoom in on the upper middle of the graph, about a quarter or an eighth of the whole area. Keep slicing in with the knife. You may have to retreat once or twice if you try to zoom in too much in one step. What you see will be very jagged and intersection points will come and go. It's best to think of your task this way: you are eliminating the area of the graph that you know does not have the intersection in it.

Keep going until you have a clear intersection point like this:



At this point, choose **Find Graph Root** from the **Manipulate** menu.



As in the case of the one-dimensional root, Theorist attempts to calculate the answer to full precision. Note that this is the same solution found the last time. By plugging these into the equation for *x*, you can find that, too.

Group Theory

This tutorial introduces the basic definitions of a group and describes:

- Creating a group of three non-commutative elements
- Using Transformation rules to explore permutation groups

Group theory is a branch of mathematics concerned with multiplying abstract objects together to make other abstract objects. The objects, called elements, can be numbers, vectors, matrices, differential operators, or rotation operators. The "multiplication" can be anything that vaguely resembles multiplication. For instance, addition over integers modulus 7 constitutes a group where the integers mod 7 are the elements and addition is the operation. Group theory is not concerned so much with the mechanics inside the elements and how the "multiply" operation is done, but is concerned more with how the elements operate on each other and the properties of the group of elements as a whole.

The minimum requirements for a group are that you have unique elements (at least two) and an operation. The operation must have:

❶ Closure—if you operate on any two elements in the group, the result has to be in the group, too.
❷ Associativity—$a(bc) = (ab)c$.
❸ Identity—there must be one element like 1 where 1 operated on a equals a for all a.
❹ Inverse—for every element, there must be an inverse element in the group (which may be itself). That is, when the element is operated on its inverse, or vice versa, the result is the identity element.

Notice that there is no rule about being commutative. By declaring operators to be M-Linear in Theorist, we can study group theory, because M-Linear operators are not necessarily commutative.

From these simple rules, a very interesting world opens up. For instance, there is only one unique group that has two elements in it. All other groups with two elements are isomorphic (i.e., they all work exactly the same). One representation of this group consists of the numbers 1 and –1 being multiplied together. 1 is the identity element and –1 is the other element which is its own inverse. It is like handling a coin: leaving the coin alone is the identity; turning it upside down is its own inverse.

One useful tool for thinking about groups is to make a multiplication table for it. This can easily be done in Theorist.

Enter the following expression:

$$\square \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} (1,2,3,4)$$

Expand it:

$$\triangle \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} (1,2,3,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{pmatrix}$$

This is a multiplication table. The top edge can be considered column headings as well as entries in the table for the row "1". Similarly, the left edge can be considered row headings as well as entries in the table for column "1". You can use this trick on any group you study in Theorist, or in fact any kinds of expressions that you want to make a multiplication table for. In this case, for instance, you can see that the numbers 1, 2, 3 and 4 do not constitute a group because it is not closed under multiplication (you get outside elements like 6 and 16).

Group of Three

Then, the question is, how many groups are there with three elements? In other words, how many different multiplication tables can you make out of the elements 1, A and B?

Be sure to type * or [space] between the A and the B.

Create the following three case theories:

$$\bigcirc\ \square\ AB = A$$

$$\bigcirc\ \square\ AB = B$$

$$\bigcirc\ \square\ AB = 1$$

Clarify the notebook and declare A and B both as M-Linear operators. We will now try to work out as many multiplication tables as we can without contradiction.

We will start by deriving what AB is. In the first case there is some element, A^{-1}, that is the inverse of A. We will apply that onto the front of the first equation, $AB = A$. Select the whole equation. Click [Apply icon] (or choose **Apply** from the **Manipulate** menu), then press the left arrow key [←], then type [space] **A^-1**. You get:

$$\triangle\ A^{-1}AB = A^{-1}A$$

Now, select it and Simplify it. Theorist neutralizes the *A*'s and yields:

$$\triangle\ B = 1$$

This is a contradiction. If it were true, then there would be only two elements in the group, and we would not have a three element group.

There is an easier way to do this. Grab the *A* on the left side of the first equation and move it over to the other side. Theorist gives you the same result, because it internally uses the same operations to produce that result.

Delete this case theory because we know it cannot be true.

By sight, we can see the same thing will happen with the second case. If you see fit, you can go through the motions to convince yourself, this time moving *B* over. Delete this case theory also because we know it cannot be true.

So we are stuck with the last case. Given that it must be true (or else it is impossible to have a group with three elements), let's derive some other facts about it. Isolate *A* to get the following.

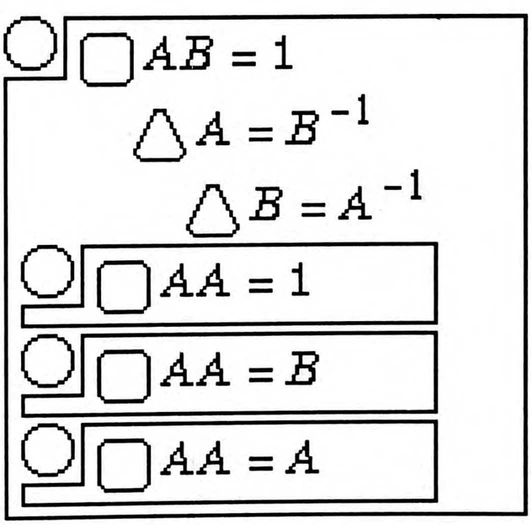
$$\triangle\ A = B^{-1}$$

We can also Isolate *B* to get the following.

$$\triangle\ B = A^{-1}$$

To make the three cases within the existing case theory, first create the assumptions, then surround each with the Case Theory command.

Now we have to figure out what *AA*, or A^2, is. In the existing case theory, make three additional cases for the three possibilities.



Let's start with the equation $AA = A$. Move an A over from the left to the right side. This yields a contradiction.

$$\triangle\ A = 1$$

Therefore, the conjecture $AA = A$ is wrong. Delete its case theory.

By moving over A in the $AA = 1$ trial theory, and substituting in $B = A^{-1}$ to change A^{-1} into B, we can see it is also wrong.

$$\triangle\ A = B$$

Delete this theory too. Substitute $B = A^{-1}$ into $AA = B$ and Simplify.

$$\triangle\ AA = A^{-1}$$

$$\triangle\ A^2 = A^{-1}$$

Then Substitute $B = A^{-1}$ into A^{-1}.

$$\triangle\ A^2 = B$$

Copy the remaining equations out of the case theories and paste them into the main part of the notebook, then delete the case theories.

From these equations, we can derive equations for B^2 and BA that equate them to individual elements of the group. Start by creating the following new assumption.

$$\square\ BA$$

Substitute $B = A^{-1}$ into it, which gives us our answer for BA.

$$\triangle\ BA = 1$$

Create a new assumption for BB.

$$\square\ BB$$

Take $A^2 = B$ and substitute it into the left of the two B's, which gives you:

You could substitute into the right B, which results in $BB = BA^2$, which is the same answer in a different form.

$$\triangle\ BB = A^2B$$

Take $B = A^{-1}$ and substitute it into the B of the right-hand side of the above equation. Simplify the result.

$$\triangle\, B^2 = A$$

Now we have enough information to make our multiplication table. Select the four equations for A^2, B^2, AB and BA *en masse* by Shift-clicking their equals signs, then choose **Transform Rule** from the **Notebook** menu's **Insert** submenu. New transformation rule propositions appear below each equation. Change each rule's pop-up menu to "Simplify."

Upon [Simplify] transform A^2 into B .

Upon [Simplify] transform B^2 into A .

Upon [Simplify] transform BA into 1.

Upon [Simplify] transform $A\,B$ into 1.

Next, enter the following expression.

$$\square \begin{pmatrix} 1 \\ A \\ B \end{pmatrix} (1, A, B)$$

Expand it to get your multiplication table:

$$\triangle \begin{pmatrix} 1 \\ A \\ B \end{pmatrix} (1, A, B) = \begin{pmatrix} 1 & A & B \\ A & B & 1 \\ B & 1 & A \end{pmatrix}$$

This describes the one unique group that has three elements, because there are no other possibilities.

Exercise

Determine how many groups there are with four elements.

Permutation Groups

Among the most interesting kinds of groups are permutation groups. A permutation is a rearrangement of elements. For instance, if we think about a list of three numbers (1, 2, 3), there are six different ways to arrange them. We can imagine each such list as an operator, rearranging the elements of another list according to its own number order. For instance, the operator (3, 2, 1) would turn all the elements backwards, so (3, 2, 1)(1, 3, 2) would be (2, 3, 1).

The identity element in this group would be (1, 2, 3). There are all sorts of little subgroups, too. For instance, the element (1, 3, 2) will simply interchange the last two elements, so it is its own inverse.

In a new notebook, make a rule to calculate the multiplication.

Upon [Simplify ▾] transform $(l, m, n)p$ into (p_l, p_m, p_n).

With this transformation rule, two lists (row vectors) of integers multiplied together turn into the result that you would get if you permuted them according to our guidelines. If this is applied to several lists multiplied together, the permutations of integers multiply against each other and eventually reduce down to a single list of three.

We can do this only because Theorist does not have any built-in way of processing row vectors that are multiplied. A row vector and a column vector are a different story, because matrix multiplication might happen, but that is not important for what we want to do.

Try it out by creating a new assumption and typing **(3,2,1)(1,3,2)** then Simplifying it.

$$\triangle\ (3,2,1)(1,3,2) = (2,3,1)$$

A problem occurs when you try to operate two identical lists together. Make a new assumption and type **(3,2,1)(3,2,1)** then Simplify it.

$$\triangle\ (3,2,1)(3,2,1) = (3,2,1)^2$$

In this case, Theorist is being too smart for its own good. We can work around this by including a power law which handles all powers iteratively. The easiest way to make it is to copy and paste the previous rule and modify it so our new rule is as follows.

Upon [Simplify ▾] transform $(l, m, n)^j$ into
$$\left(\left[(l, m, n)^{j-1}\right]_l, \left[(l, m, n)^{j-1}\right]_m, \left[(l, m, n)^{j-1}\right]_n\right).$$

Now, we can go back and simplify the square we had:

$$\triangle\ (3,2,1)(3,2,1) = (1,2,3)$$

That's six semi-colons ; and six commas ,

Now, let's make a multiplication table for this. Make a new assumption and type **(;;;;;;)(,,,,,,)**. Select the whole product of both matrices. Click Select In twice. Type **(9,9,9)**, which gives you:

$$\square \begin{pmatrix}[9,9,9]\\ [9,9,9]\\ [9,9,9]\\ [9,9,9]\\ [9,9,9]\\ [9,9,9]\end{pmatrix}([9,9,9],[9,9,9],[9,9,9],[9,9,9],[9,9,9],[9,9,9])$$

Now, we are going to change all of those 9's to vectors. Select the six positions as shown:

$$\square \begin{pmatrix}[9,9,9]\\ [9,9,9]\\ [9,9,9]\\ [9,9,9]\\ [9,9,9]\\ [9,9,9]\end{pmatrix}([9,9,9],[9,9,9],[9,9,9],[9,9,9],[9,9,9],[9,9,9])$$

Then type **1** [tab] **2** [tab] **3**. Next, select the six positions as shown:

$$\square \begin{pmatrix}[1,2,3]\\ [3,1,2]\\ [2,3,1]\\ [9,9,9]\\ [9,9,9]\\ [9,9,9]\end{pmatrix}([1,2,3],[3,1,2],[2,3,1],[9,9,9],[9,9,9],[9,9,9])$$

Then, type **3** [tab] **2** [tab] **1**. Select the product of both matrices and Expand.

$$\triangle \begin{pmatrix}[1,2,3]\\ [3,1,2]\\ [2,3,1]\\ [3,2,1]\\ [1,3,2]\\ [2,1,3]\end{pmatrix}\cdots\cdots = \begin{pmatrix}[1,2,3] & [3,1,2] & [2,3,1] & [3,2,1] & [1,3,2] & [2,1,3]\\ [3,1,2] & [2,3,1] & [1,2,3] & [1,3,2] & [2,1,3] & [3,2,1]\\ [2,3,1] & [1,2,3] & [3,1,2] & [2,1,3] & [3,2,1] & [1,3,2]\\ [3,2,1] & [2,1,3] & [1,3,2] & [1,2,3] & [2,3,1] & [3,1,2]\\ [1,3,2] & [3,2,1] & [2,1,3] & [3,1,2] & [1,2,3] & [2,3,1]\\ [2,1,3] & [1,3,2] & [3,2,1] & [2,3,1] & [3,1,2] & [1,2,3]\end{pmatrix}$$

Notice that since we put the identity element in the first position, the top row serves as column headings, and the left column serves as row headings.

Looking at this table we see four subgroups in it. The first is the upper left quarter of the table, a square of nine cells, whose elements are all the set (1, 2, 3) in various stages of rotation. This is isomorphic to the group of three from the last section (because it is a group with three elements in it).

The second group is found at the intersections of the first and fourth rows and columns, with the elements (1, 2, 3) and (3, 2, 1). You can think of this group as the one that rearranges the elements backwards and forwards, or you can think of it as the one that interchanges the first and last element. This is of course isomorphic to the simple group of two discussed earlier.

Similarly, the third subgroup can be seen in the first and fifth rows and columns, and the last subgroup can be seen in the first and last rows and columns. These also interchange two elements. This illustrates a very interesting area of group theory, the study of subgroups.

These examples have been relatively simple; most people who do group theory have this permutation group memorized. Group theory is used for a wide variety of things such as Rubik's Cube, rotations in space, and in a wide variety of areas in physics concerning quantum mechanics and subatomic and sub-nuclear particles.

Differential Calculus

LIMITS OF FUNCTIONS

You can use Theorist to estimate the limit of a function at a particular point either graphically or numerically as the following example illustrates. In general, it is best to create a graph of the function you are trying to find the limits of before proceeding with numerical evaluations.

Below, we will examine the limit as $x \to 0$ of the following expression:

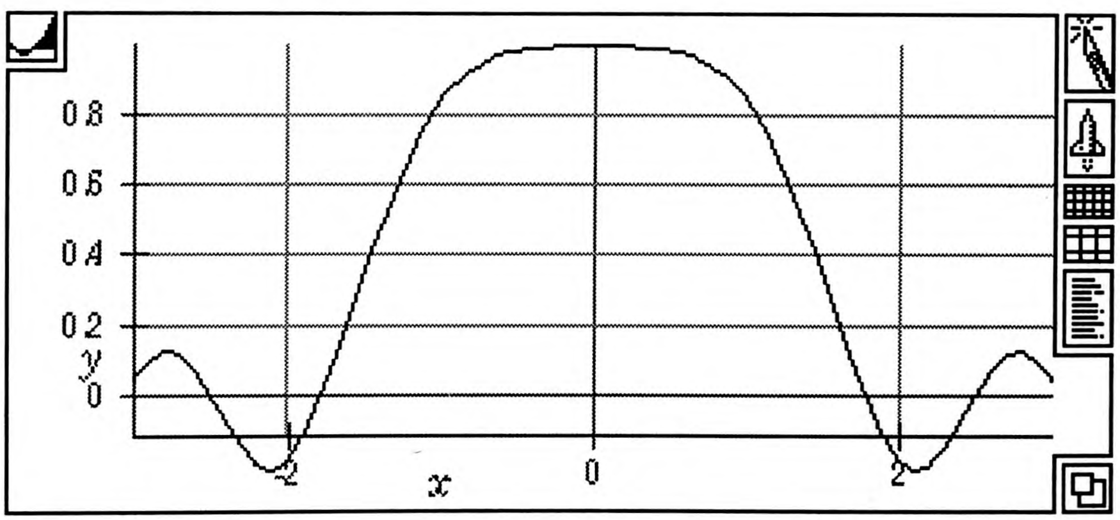
$$\frac{\sin(x^2)}{x^2}$$

Graphical Solution

Let us examine the expression graphically using Theorist. Type [return] **y=sin(x^2)/x^2**.

$$\square\ y = \frac{\sin(x^2)}{x^2}$$

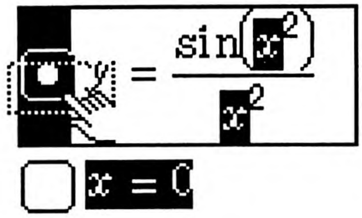
Select the entire expression, then click the graph button or choose **Graph▸ y=f(x)▸Linear**. The following graph is displayed:



Examining the graph near $x = 0$ using the Knife tool, we can see that the limit might be 1. Note that even at higher resolutions, the graph does not show the discontinuity ("hole") at zero (see below), which illustrates the limits of accuracy of machine-generated graphs.

Numerical Solution

Type [return] **x=0**, then select this equation and drag it over the assumption icon of the original expression.



Note the question mark in the result.

$$\triangle\ y = ? \quad \textit{Substitute}$$

Theorist returns a question mark when it is unable to calculate a value for an expression. To examine this behavior more closely, turn Auto Simplify off, then perform the same substitution again.

$$\triangle\ y = \frac{\sin(0^2)}{0^2} \quad \textit{Substitute}$$

Simplify the numerator.

$$\triangle\ y = \frac{0}{0^2} \quad \textit{Simplify}$$

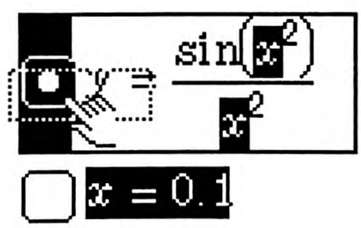
Simplify the denominator of this conclusion, which produces the indeterminate form 0/0 in the final conclusion.

$$\triangle\ y = \frac{0}{0} \quad \textit{Simplify}$$

Turn Auto Simplify back on.

Approximating the Limit Numerically

Type [return] **x=0.1** to let $x = 0.1$. Substitute this value for x into the original assumption defining y.



Select the right-hand side of the resulting expression,

$$\triangle\ y = 100\sin(0.01) \quad \textit{Substitute}$$

...then click or choose **Calculate** from the **Manipulate** menu.

$$\Delta y = 0.99998$$ *Calculate*

Exercise

Repeat for other values of x closer to zero (e.g., 0.01, 0.001, 0.0001, etc.). You will see the limit approach 1 as x is made smaller.

In order to confirm that you have indeed found an approximation for the limit, approach zero from the other direction. Substitute $x = -0.1$, $x = -0.01$, etc.

SLOPES AND TANGENTS

Theorist can be used to calculate the slope m of a line tangent to the graph of a function f at a given point $P(a, f(a))$, using the definition of the derivative.

Slopes

The definition for the slope, m, at $x = a$ is given by the following equation.

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If the limit in this definition does not exist, the slope of the tangent line at $P(a, f(a))$ is undefined.

Let us determine a slope of the following function at point x:

$$f(x) = x^2 + \frac{1}{x}$$

Using the definition for the slope and extending the example on limits in the previous tutorial, we can determine the slope by entering an expression for m into Theorist. Type [return] **m=(((x+h)^2** [esc] **+1/(x+h))-(x^2** [esc] **+1/x))/h**.

$$\square\; m = \frac{\left([x+h]^2 + \frac{1}{x+h}\right) - \left(x^2 + \frac{1}{x}\right)}{h}$$

Clarify the notebook. Declare m and h as user-defined variables in the dialog, by clicking on the **User Defined** buttons in the dialogs or by pressing [return] each time.

Now let's find the slope at the point where x equals 2. First, type [return] **x=2**, and then substitute x into the equation that defines m.

$$\triangle\; m = \frac{(h+2)^2 + \frac{1}{h+2} - \frac{9}{2}}{h} \quad \textit{Substitute}$$

Expand the right side of the resulting expression twice.

$$\triangle\; m = -\frac{9}{2}\frac{1}{h} + \frac{1}{(h+2)h} + \frac{h^2+4h+4}{h} \qquad \textit{Expand}$$

$$\triangle\; m = -\frac{1}{2}\frac{1}{h+2} + h + 4 \qquad \textit{Expand}$$

Because the slope formula is continuous at h=0, the limit for m can be calculated directly. Type [return] **h=0**, and substitute it into the expression for m. The final result is:

$$\triangle\; m = \frac{15}{4} \qquad \textit{Substitute}$$

This is the slope of the line tangent to the point at which x equals 2.

Tangent Lines

Knowing the slope, we would like to find the equation of the tangent line passing through the point $(2, f(2))$. The Point-Slope form for a line with a slope m through a point (x_1, y_1) is:

$$y - y_1 = m(x - x_1)$$

To perform a multiple substitution, drag to the rounded square icon for the equation statement.

Using this formula, we can find an equation that describes the tangent line of the original function at x = 2. Type [return] **y=x^2** [esc] **+1/x.** Substitute the existing value for x into the equation for y. The result is y_1 as given by the following expression:

$$\triangle\; y = \frac{9}{2} \qquad \textit{Substitute}$$

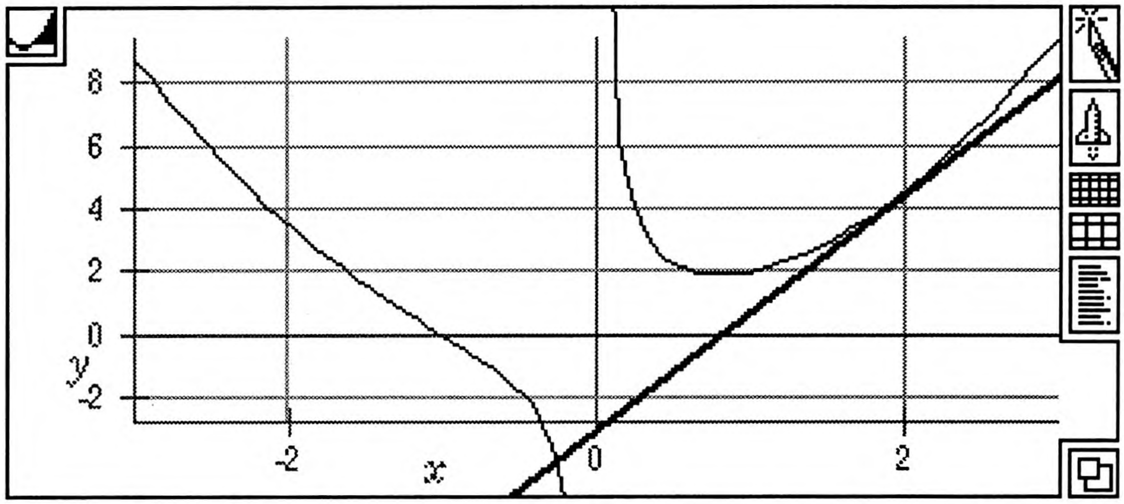
Type [return] **y-9/2** [esc] **=m(x-2)**, the equation for the tangent line. Then substitute the previous equation for m into the equation for the tangent line.

$$\triangle\; y - \frac{9}{2} = \frac{15}{4}(x-2) \qquad \textit{Substitute}$$

Solve this equation for y by selecting y, then choosing **Isolate** from the **Manipulate** menu. Expand the right-hand side of the result, then select y on the left-hand side and type **y'**. This creates a new expression.

$$\square\; y' = \frac{15}{4}x - 3$$

We are now ready to graph our original function and a line tangent to it at x = 2. Select the original expression that defines y, then click [graph icon] or choose **Graph▸y=f(x)▸Linear**. Declare y' as a variable if you have not previously done so, and declare π predefined. Select the y' equation, which defines the tangent line, then choose **Graph▸Additional▸Add Line Plot**. Press [return] to accept the definitions for the axis variables. You may open the resulting graph's details and change the added line plot style and/or color.



You may zoom in on the region near $x = 2$ for a closer look. This is the only point where the graph and the curve touch.

DERIVATIVES

Derivatives of functions may be determined two different ways in Theorist. The first technique uses the differential operator *d*, and the second uses the partial derivative operator. This chapter describes differentiation and a later chapter describes partial differentiation.

Let's differentiate the following equation:

$$y = 3x^3+2x+\sin(x)$$

Enter this equation by typing **y=3x^3** [esc] **+2x+sin(x)**:

$$\square\ y = 3x^3 + 2x + \sin(x)$$

In the New Notebook, *d* is predefined as the differential operator.

Now set up a template for the derivative by pressing the [return] key and typing **(d*y)/d*x**. Clarify the notebook if you wish to remove the parentheses.

$$\square\ \frac{dy}{dx}$$

Substituting the equation that defines *y* into the second statement will differentiate it. This produces the following conclusion:

$$\triangle\ \frac{dy}{dx} = \left(2dx + \cos[x]dx + 9x^2dx\right)\frac{1}{dx} \quad \textit{Substitute}$$

This expression includes *dx* terms which will cancel out. Select the right-hand side and click the [Expand] palette button (or choose **Expand** from the **Manipulate** menu):

$$\triangle\ \frac{dy}{dx} = \cos(x) + 9x^2 + 2 \quad \textit{Expand}$$

This is the equation for the derivative.

MAXIMUM AND MINIMUM

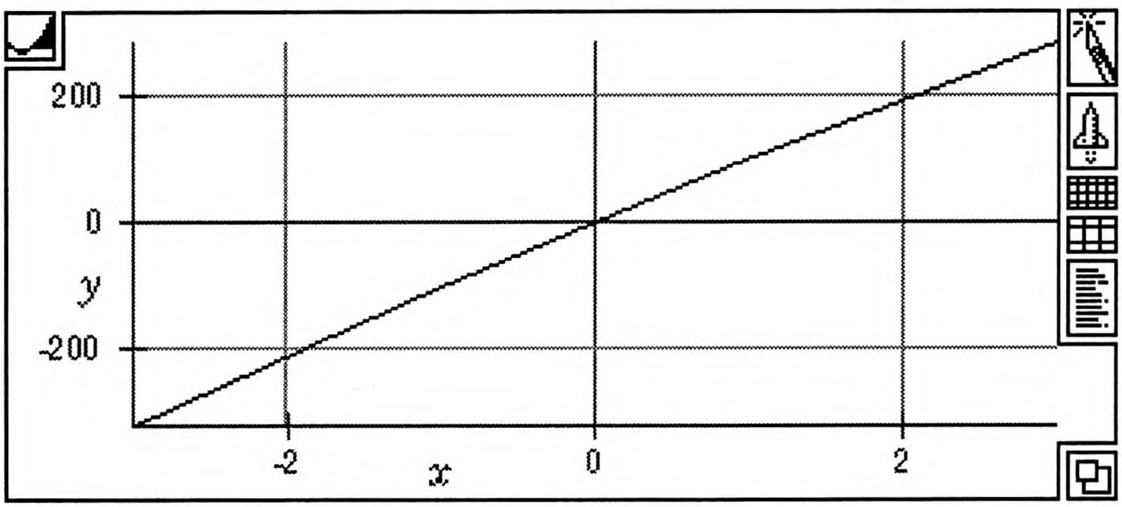
This tutorial describes how to find the maximum value of a function by using the partial differential operator and includes examples of the Substitute and Isolate manipulations.

Type **y=x*(100-2*x)** to make a function of x:

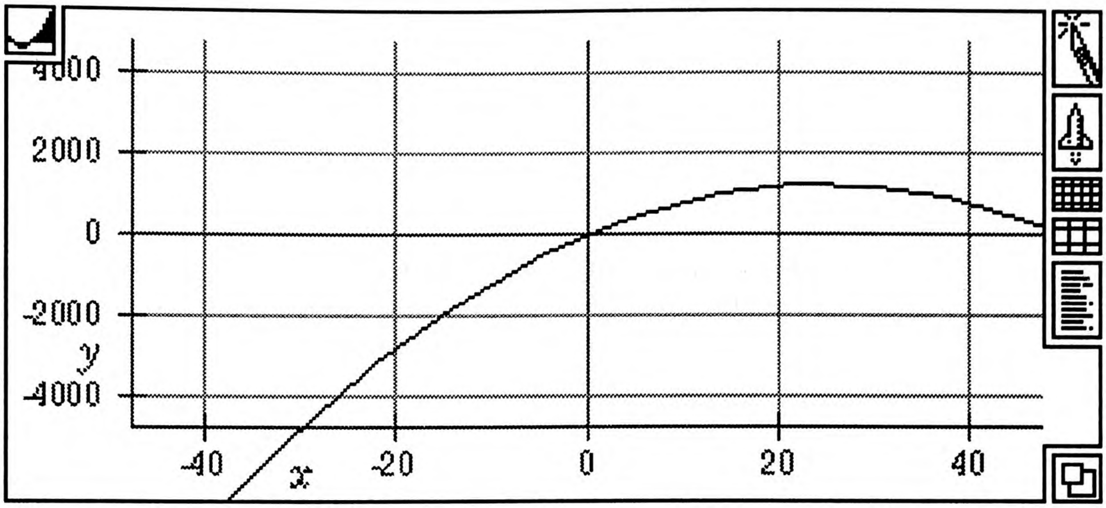
$$y = x(100-2x)$$

You want to know at what value of x your function obtains its maximum or minimum and what is the maximum or minimum value of your function at those points.

To get a visual idea of the function, click (or choose **Graph▸ y = f(x)▸Linear**) to create a two-dimensional graph.



The default display does not provide very much useful information. Click on the Rocketship button several times to zoom out and see what is going on.



You want to find the x and y values at the peak. This is the place where the derivative with respect to x of y is zero. It appears that the peak is near $x = 20$ and $y = 1000$.

Typing hint for the next three steps: **Derivative(x)** [esc] [space] **y=0**.

Make a new assumption. Click $\frac{\partial}{\partial a}$ on the function palette. This gives you:

$$\square \frac{\partial}{\partial ?}$$

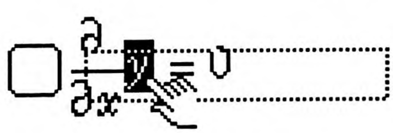
The highlighted question mark is your current insertion point. Type **x** [esc] **y=0**.

$$\square \frac{\partial}{\partial x} y = 0$$

Click on the equal sign in your first equation to select it.

$$\boxed{\bullet}\ y = x(100 - 2x)$$

Now hold down the [⌘] key (Macintosh) or [Ctrl] key (Windows) and drag the highlighted equation over the y in the $\frac{\partial}{\partial x} y = 0$ expression:



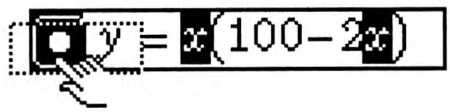
Theorist performs the Substitute manipulation and differentiates your function.

$$\triangle\ -4x + 100 = 0 \quad \textit{Substitute}$$

For the next step you want to solve this equation. Select x and click (or choose **Isolate** from the **Manipulate** menu).

$$\triangle\ x = 25 \quad \textit{Isolate}$$

This is the point at which the derivative of $y = x\,(100 - 2x)$ equals zero. Therefore, for $x = 25$, the function y reaches its maximum (as we suspected from the graph). To find the corresponding value of y, substitute $x = 25$ into $y = x\,(100 - 2x)$. You do this in much the same way as in the previous substitution, except you drag over the assumption icon (the square) instead of the x (because you want to substitute for both x's).



This results in:

$$\triangle\ y = 1250 \quad \textit{Substitute}$$

PARTIAL DIFFERENTIATION

Partial derivatives of an expression may be determined by applying Theorist's partial derivative operator to the expression.

Let us find the partial derivative with respect to x and y of the following expression for z, a function of x and y:

$$z = x^2y + \cos(xy)$$

Type **z=x^2** [esc] **y+cos(x*y)**. Make a new assumption by pressing [return]. Click the palette's partial derivative op button, $\frac{\partial}{\partial a}$, or type **Partial(**.

If you typed it, there are parentheses surround its question mark.

$$\square\frac{\partial}{\partial ?}$$

Type **x** to replace the selected question mark, press [esc] to move the cursor, and type **z**. Substitute the equation for z into this partial derivative expression by dragging.

$$\triangle\ \frac{\partial}{\partial x}z = -\left(x\frac{\partial}{\partial x}y + y\right)\sin(xy) + 2xy + x^2\frac{\partial}{\partial x}y \qquad \textit{Substitute}$$

This is probably not exactly the result you expected. There are two extra differentiation terms in the result because Theorist does not assume x and y are independent of each other. We can specify this relationship by choosing **Independence Decl.** from the **Notebook** menu's **Insert** submenu, then typing **x,y**.

The variables (x, y) are independent of [each other ▼].

Theorist treats y as if it were a constant.

Now repeat the substitution you did previously.

$$\triangle\ \frac{\partial}{\partial x}z = -y\sin(xy) + 2xy \qquad \textit{Substitute}$$

Now let us find the partial derivative of z with respect to y. Change x to y in the derivative assumption you entered. Theorist automatically remanipulates the conclusions.

$$\triangle \frac{\partial}{\partial y} z = -x \sin(xy) + x^2 \quad \textit{Substitute}$$

You have now found both partial derivatives for the function z.

Integral Calculus

Simple Indefinite Integrals

This tutorial shows how Theorist can perform indefinite integration with and without constants of integration. Indefinite integrals are also called symbolic integrals, since they do not produce a numeric result.

Click the ∫a palette button and type **cos(x)** [tab] **x**.

$$\square \int \cos(x)\,dx$$

Click on the integral sign to select the expression. Click the palette button (or choose **Simplify** from the **Manipulate** menu) and, after declaring *d,* you get the following conclusion:

$$\triangle \int \cos(x)\,dx = \sin(x) \quad \textit{Simplify}$$

Constants of Integration

When you evaluate an indefinite integral, the result is ambiguous, since you could add any constant value to the expression you obtained, and the resulting expression would be equally correct. (You can see this for yourself: try taking the derivative of the sine function plus a couple of different constants.) It is apparent that there is an infinity of possible solutions, since the constant could be anything. Therefore, a symbolic constant is often added to the simple solution to provide a general solution. The usual convention is to name the constant *C.* Sometimes, the constant is a nuisance and people leave it out.

Auto Casing is off by default in the New Notebook.

In Theorist, you can decide whether or not you want this constant to be generated. When the Auto Casing preference is on, Theorist generates an arbitrary constant when it does an integration. If Auto Casing is off, there will be no arbitrary constant.

For example, type [return] **$x^2** [esc] **d** [space] **x**, click the integral sign or the E=mc palette button, and Simplify. With Auto Casing off you get:

$$\square \int x^2\,dx$$

$$\triangle \int x^2\,dx = \frac{1}{3}x^3 \quad \textit{Simplify}$$

Choose **Auto Casing** from the **Manipulate** menu's **Preferences** sub-menu (to turn it on) and Simplify again to get:

$$\triangle \int x^2 dx = \frac{1}{3}x^3 + c_{100} \quad \textit{Simplify}$$

Theorist's default name for the constant is the letter *c* with a numerical subscript. Each time such a constant is generated by Theorist, it generates a new subscript for you. You can change this number, and you can choose a name other than *c*, in a special dialog. Choose **Arbitrary Constants...** from the **Manipulate** menu's **Preferences** submenu. The name you choose must be declared as a constant.

INTEGRATION BY PARTS

This tutorial describes a technique for evaluating integrals using a powerful feature of Theorist, Integration by Parts. Integration by parts is used when your integrand consists of two parts multiplied together, and you know how to integrate one and how to differentiate the other. Since differentiating is usually easy, this technique is useful for integrals that can not be solved any other way. Integration by parts does not always solve your integral, since the end result includes another integral. The formula for the integration by parts technique shows this:

u and *v* are functions of the same variable; in this case we'll be using *x*.

$$\int u\,dv = uv - \int v\,du$$

Enter the following integral by typing **$(x*e^x** esc ***d*x):'p:2'p**. Clarify to remove unnecessary parentheses.

$$\Box \int_{\pi}^{2\pi} x e^{x} dx$$

Select the expression (click the integral sign or the palette button) then click the palette button (or choose **Simplify** from the **Manipulate** menu).

$$\triangle \int_{\pi}^{2\pi} x e^{x} dx \;=\; \Big[\int e^{x} x\, dx \Big]_{x=\pi}^{x=2\pi}$$

Theorist took the limits off the integral, but could not integrate it. This expression is a candidate for integration by parts. First we have to decide which part to integrate and which part to differentiate.

The e^x stays the same if we integrate it or differentiate it. The x gets more complicated if we integrate it, but it turns into 1 if we differentiate it We should select the part to integrate, so we want to select e^x and dx.

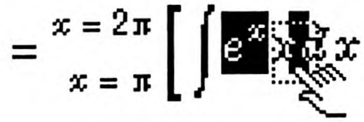
Just don't try *too* hard because it can't be done.

Try to make a multiple selection of e^x and dx (as shown below) by dragging and holding the shift key.

$$\Big[\int e^{x} x\, dx \Big]_{x=\pi}^{x=2\pi}$$

Theorist does not allow these terms to be multiple selections because selecting the dx means selecting the product of all four things.

Therefore, we will use the Commute manipulation, by hand (with the ⌘ or Ctrl key). Grab the e^x and move it to between the x and dx, like so:



This gives us:

$$\triangle \int_{\pi}^{2\pi} x e^{x} dx = \left[\int x e^{x} dx\right]_{x=\pi}^{x=2\pi}$$

Choose **Auto Casing** from the **Manipulate** menu's **Preferences** submenu (to turn it on). Select $e^x dx$ and choose [∫udv] from the [hammer] pop-up palette (or choose **Int. by Parts** from the **Manipulate▸Other** submenu).

$$\triangle \cdots\cdots = \left[-e^{x} - c_{106}x + \left(e^{x} + c_{106}\right)x + c_{109}\right]_{x=\pi}^{x=2\pi}$$

The expression to the left of the equal sign has been collapsed for space. You can leave it exposed if you like.

(In your answer, the arbitrary constant c may have different subscripted numbers; this is fine.)

Select the expression to the right of the equals sign and click [Expand icon] (or choose **Expand** from the **Manipulate** menu), which produces the final answer.

$$\triangle \left[\int e^{x} x\, dx\right]_{x=\pi}^{x=2\pi} = 2\pi e^{2\pi} - e^{2\pi} - \pi e^{\pi} + e^{\pi}$$

Alternate Method

There is another way of separating e^x and dx to integrate by parts, which we will do.

Copy the original integral expression to the clipboard. Make a new notebook and paste it in. Turn on Auto Casing. Select and Simplify the integral, as in the first step of the other method. Next, select dx alone and Simplify, which gives us a slightly different result.

$$\triangle \int_{\pi}^{2\pi} x e^{x} dx = \left[\int e^{x} x(dx)\right]_{x=\pi}^{x=2\pi}$$

Note the parentheses. Now the integral is the product of three things, the last of which is dx. Make a multiple selection of the e^x and dx; this time Theorist allows it.

Give the Integrate by Parts command and Expand as before, to get the same final answer.

$$\Delta_{x=\pi}^{x=2\pi}\left[\int e^x x\,dx\right] = 2\pi e^{2\pi} - e^{2\pi} - \pi e^{\pi} + e^{\pi}$$

Exercise Try integrating $\int x^2 \cos(x)\,dx$ by parts. (Integrate the $\cos(x)\,dx$ part again.) Hint: you must do it twice.

Exercise Try integrating $\int x^7 \cos(x)\,dx$ by parts. It may be tedious.

Exercise Try integrating $\int e^x \cos(x)\,dx$ by integrating by parts on $\cos(x)\,dx$. Hint: after doing it twice, use some algebra to solve for the integral.

Exercise Try the previous exercise, integrating $e^x dx$ the second time. Integration by parts can lead you in circles if you are not careful.

Exercise Try integrating $\int \ln(x)\,dx$, but don't cheat by simplifying it. Instead, integrate by parts *just* on the dx part. Although it does not seem like it should work, it does. Do it with Auto Simplify off to watch what happens step-by-step.

Exercise Find out if arbitrary constants interfere. Try doing an integration by parts with Auto Casing on.

INTEGRATION BY TRANSFORMATION AND SUBSTITUTION

Theorist can also evaluate integrals that normally require a substitution to reduce the problem to something simpler that Theorist (or we) knows how to evaluate. For example, consider the following integral.

$$\int \cos^3(x)\,dx$$

The Theorist form for this expression is slightly different.

Type **$(cos(x))^3** esc **d*x**.

$$\square \int (\cos[x])^3 d\,x$$

Click on the integral sign to select the entire integral, then click (or choose **Simplify** from the **Manipulate** menu). You should see a brief flash below the original expression because Theorist does not know how to evaluate this integral—yet!

This integral can be solved by reducing the order of this expression from a cubic to a quadratic. Choose **Manipulate▸Preferences▸Auto Simplify** to turn it off. Select $(\cos[x])^3$ and click (or choose **Collect** from the **Manipulate** menu).

$$\triangle \int (\cos[x])^3 d\,x \quad = \int (\cos[x]\cos[x]\cos[x])\,d\,x \qquad \textit{Collect}$$

Select "cos[x] cos[x]" (just two of the three) and Simplify. Theorist creates the following equality.

$$\triangle \int (\cos[x])^3 d\,x \quad = \int ([\cos\{x\}]^2 \cos[x])\,d\,x \qquad \textit{Simplify}$$

Trigonometric Identities

Now we need to transform the $(\cos\{x\})^2$ term using the trigonometric identity $(\cos[x])^2 = 1 - (\sin[x])^2$, which is included as a transformation rule in your notebook's Declarations. (It is collapsed in the Standard Rules bundle's Trigonometry bundle's second Transform bundle.)

Notebooks based on the New Notebook file include this and other transformation rules.

Upon [Transform ▾] transform $(\cos[x])^2$ into $1-(\sin[x])^2$.

Select $(\cos[x])^2$ in the integral and click the palette button (or choose **Transform** from the **Manipulate** menu). Since there is another rule for $(\cos[x])^2$ in the notebook's declarations, a dialog asks you to choose your desired result. Click on the $1-(\sin[x])^2$ result.

$$\triangle \int (\cos[x])^3 dx = \int \left(\left[1-(\sin(x))^2\right]\cos[x]\right) dx \qquad \textit{Transform}$$

Select the whole integral (from the right side of the equal sign) and click (or choose **Expand** from the **Manipulate** menu).

$$\triangle \int (\cos[x])^3 dx = \int -\cos(x)(\sin[x])^2 dx + \cos(x) dx \qquad \textit{Expand}$$

Trigonometric Substitution

The expression has now been partly integrated, and the remaining portion of the integral is in a form that is easier to handle using a trigonometric substitution. We will use the substitution $u=\sin(x)$.

Choose **Manipulate▸Preferences▸Auto Simplify** to turn it back on. Type [return] **u=sin(x)** and Clarify to declare u as a variable. Select x and click (or choose **Isolate** from the **Manipulate** menu).

$$\triangle\ x = \arcsin(u) \qquad \textit{Isolate}$$

The arcsin equation should be selected. (If not, click its equal sign.) Press and hold [shift] and click the integral's ∫ sign so you have a multiple selection. Choose from the pop-up (or choose **Substitute** from the **Manipulate▸Other** submenu).

$$\triangle \int (\cos[x])^3 dx = -\frac{1}{3}u^3 + u \qquad \textit{Substitute}$$

Notice how x was substituted just inside the right-side integral (not the whole equation), as you selected.

Undoing the Substitution

The last task is to substitute the original definition for u into the result of the integration. Select the definition for u, then drag it into the u on to the right side as shown below.

Be sure to substitute u not $\sin(x)$.

$$\triangle \int (\cos[x])^3 dx = \sin(x) - \frac{1}{3}u^3 \qquad \textit{Substitute}$$

$\square\ u = \sin(x)$

Finally, we see that

$$\triangle \int (\cos[x])^3 dx = -\frac{1}{3}(\sin[x])^3 + \sin(x) \quad \textit{Substitute}$$

It's often a good idea to check your work. If you would like to prove the answer is correct, differentiate it. (You will have to Transform the $(\sin[x])^2$.

Simple Definite Integral

For this tutorial, we will use only the palette to enter expressions. Make sure you start with the palette in the functions mode (indicated by the big $f(x)$ on the left). Because Theorist automatically toggles the palette, you will notice it switches between the variables and functions palettes as you go along.

Click the definite integral $\int_b^a$ palette button.

$$\square\int_{?}^{?} ?\, d?$$

The highlighted question mark is the current selection. Replace it by clicking x from the variables palette, then click a^c, then 2, then $X_{\square}^{2}$, and finally x to complete the expression to be integrated.

$$\square\int_{?}^{?} x^2\, dx$$

Click $X_{\square}^{2}$ to move the selection to the next ?, which is for the lower limit. Switch to the functions palette and choose a-b from the a+b pop-up, then click 1. Click $X_{\square}^{2}$ to move the selection to the next ?, which is for the upper limit. Click 2.

$$\square\int_{-1}^{2} x^2\, dx$$

Now select the whole expression by clicking on the integral sign, then click (or choose **Simplify** from the **Manipulate** menu). Theorist then proceeds with the solution. Since Theorist's Auto Simplify option is on, it jumps to the final simplification.

$$\triangle\int_{-1}^{2} x^2\, dx = 3 \quad \textit{Simplify}$$

Your answer is 3. Piece of cake!

Exercise

Delete the answer and turn the Auto Simplify preference off. Evaluate the integral, and watch the steps Theorist goes through. Simplify until the above result is reached.

Yet Another Integral

Create a new notebook and type **Integral(sqrt(tan(x))*d*x)**.

$$\Box \int \sqrt{\tan(x)}\,dx$$

Type [return] **y=sqrt(tan(x))** for a substitution.

$$\Box\ y = \sqrt{\tan(x)}$$

Turn it inside out by isolating *x:*

$$\triangle\ x = \arctan\left(y^2\right)$$

Substitute this equation for *x* into the integral you entered to get the following.

$$\triangle \int \sqrt{\tan(x)}\,dx = 2\left(\int \frac{y^2}{y^4+1}\,dy\right)$$

Rational expressions are also known as rational functions and sometimes referred to as rational fractions.

This is a rational expression (a fraction with polynomials in the numerator and denominator). Expand it, and reply Yes to all resulting dialogs to get a large answer.

$$\triangle \int \sqrt{\tan(x)}\,dx = 2\left(-\frac{1}{8}\sqrt{2}\,i - \frac{1}{8}\sqrt{2}\right)(i\cdots\cdots$$

Note this has *i*'s representing the square root of –1. Expand again to get rid of all of that:

$$\triangle \int \sqrt{\tan(x)}\,dx = \frac{1}{2}\sqrt{2}\arctan\left(\sqrt{2}\,y+1\right)+\frac{1}{2}\sqrt{2}\cdots\cdots$$

But notice how it is in the variable *y* instead of *x*. We must Substitute again. Take the equation for *y* you entered and substitute it into our latest equation to get our final solution.

$$\triangle\cdots\cdots = \frac{1}{2}\sqrt{2}\arctan\left(\sqrt{2}\sqrt{\tan[x]}+1\right)+\cdots\cdots\cdots\cdots$$

Now, you can use remanipulation to see if the same tricks work on similar integrals. Make sure the **Always ReManipulate** item is checked in the **Manipulate** menu. Your notebook should have two assumptions (all the rest are conclusions), each with an occurrence of the name "tan." Select them by double clicking one name and then Shift double clicking the other.

Type **sin**. The remanipulation ripples through but notice expanding the integral does not work; there is a square root sign that does not go away easily. You end up with something that is still an integral.

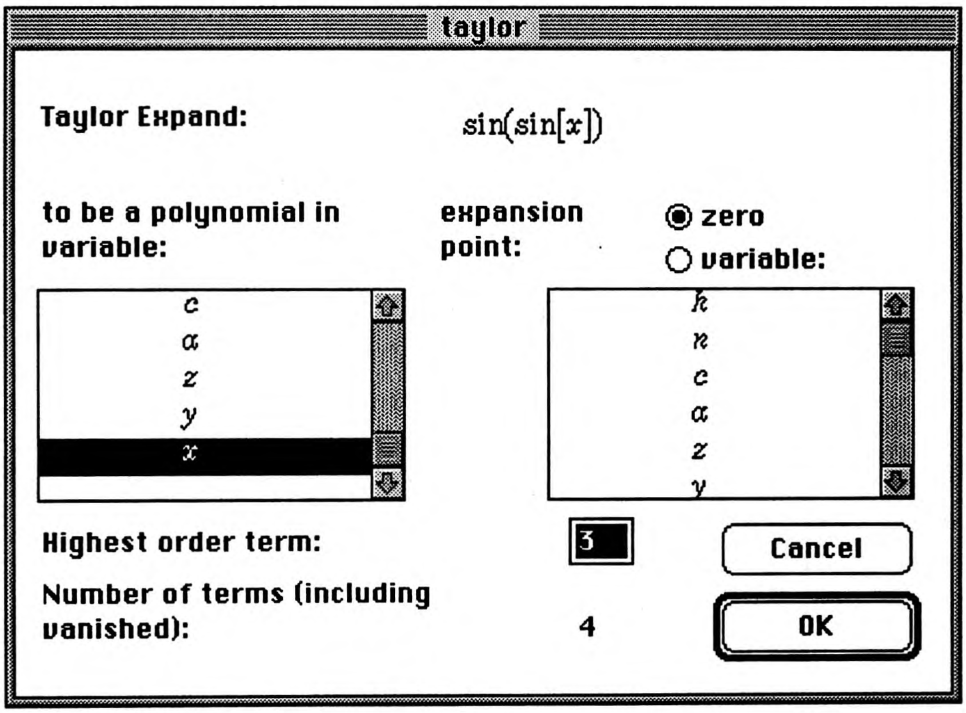
Select the sin names and type **cot**. Cotangent is similar enough to tangent that it should work. You may have to declare arccot but other than that it should all re-derive in a similar way. Note, however, that the answer contains a mixture of arctans and cots. Some of the tangents or cotangents come from the original problem, some of them come from solving the rational fraction. In fact, it would be possible to change the way these integrals are solved so that we end up with arccotangent functions, but tan is more standardized, thus the arctangents.

Taylor Series

This tutorial describes how to create a Taylor Series approximation of a smooth function. The tutorial presents a discussion of:

- The Taylor Series manipulation and its dialog box
- Graphically comparing a Taylor series to the original function

In a new notebook, type **y=sin(sin(x))**. Select the right-hand side of the equation and choose from the pop-up (or choose **Taylor Series** from the **Manipulate** menu's **Other** submenu) and the following dialog appears on the screen.



Click on the **OK** button and Theorist gives you the answer.

$$y = -\frac{1}{3}x^3 + x \quad \textit{Taylor}$$

As you can see, the expansion has two terms and not four as the dialog said. That's because the other two terms vanish at the expansion point, $x = 0$.

Select sin(sin(x)) from the first equation and do another Taylor Series command, but do not click **OK** in the dialog yet. For a better approximation you need more terms in the expansion, which you can get by changing the number in the Highest order term box in the dialog.

Change it to 7 and click on the **OK** button and Theorist crunches through computations to give you the final result, with the highest order 7:

$$\triangle\ y = -\frac{8}{315}x^7 + \frac{1}{10}x^5 - \frac{1}{3}x^3 + x \qquad \textit{Taylor}$$

To see how good the approximations are, you can graph them on the same graph. To get two graphs on the same axis, first change your equations so that the first Taylor series (up to x^3) is y', and the second Taylor series (up to x^7) is y''. (Theorist creates new assumptions.) Clarify the notebook and declare them as variables.

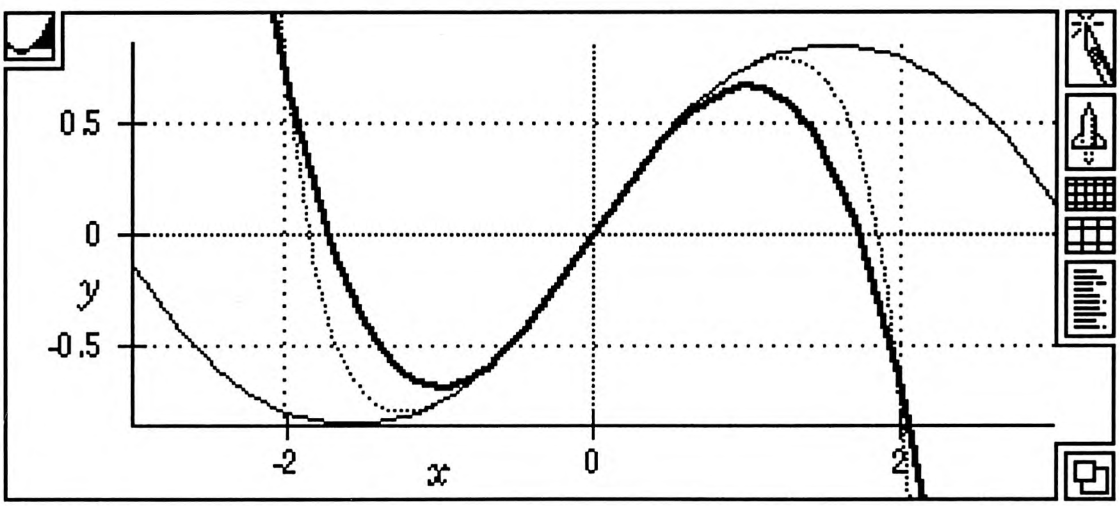
$$\triangle\ y = -\frac{1}{3}x^3 + x \qquad \textit{Taylor}$$

$$\square\ y' = -\frac{1}{3}x^3 + x$$

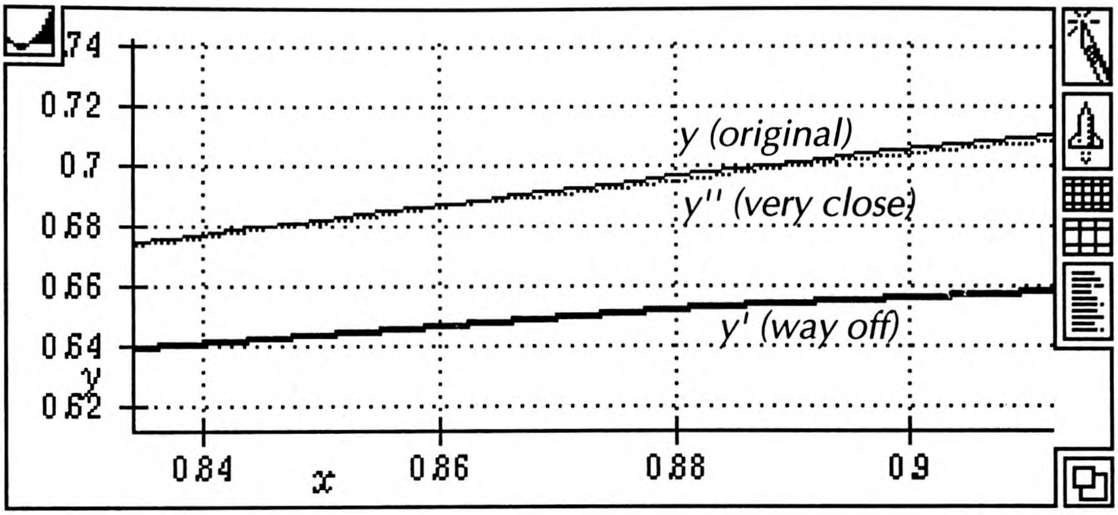
$$\triangle\ y = -\frac{8}{315}x^7 + \frac{1}{10}x^5 - \frac{1}{3}x^3 + x \qquad \textit{Taylor}$$

$$\square\ y'' = -\frac{8}{315}x^7 + \frac{1}{10}x^5 - \frac{1}{3}x^3 + x$$

To graph $y = \sin(\sin[x])$, click on the equal sign in that equation and click (or choose **Graph▸Linear▸y = f(x)**). To add a plot of the y' equation, select it and choose **Graph▸Additional▸Add Line Plot**. (If Theorist presents a dialog, click **OK**.) Repeat for the y'' equation.



The thin line represents $y = \sin(\sin[x])$. The thick and dotted lines stand for the approximations which look good near $x = 0$, but get worse as you move away. From here, it looks like we did not gain very much for our two extra terms, but if you zoom in with the knife, you can see how much better the longer approximation is.


0.74
0.72
0.7
0.68
0.66
0.64
0.62
y
y (original)
y'' (very close)
y' (way off)
0.84
0.86
0.88
0.9
x

MULTIPLE INTEGRALS

Many people use many different kinds of notation to represent multiple integrals. Theorist needs everything spelled out explicitly. You must have one integral nested entirely within the other. Each must have its differential inside it, but not inside any deeper integral.

The first integral we will evaluate is the following:

$$\int_0^1 \left(\int_0^y \sin[x+y]dx \right) dy$$

Note that the inner limits can depend upon the outer variables, but not the other way around. In reality, the limits define a piece of surface (or space, if a triple integral, or a 17-dimensional region if a 17-variable integral) that the integral is to be taken over. The integral above is fairly simple, taken over a right-triangular area of 1 unit on each side. In more complicated situations, the upper and lower limits must trace out the upper and lower edge of the domain of integration. Integration domains with holes, broken pieces or other discontinuities will require more work (perhaps using more than one integral for different pieces, or multiplying in a function that is zero or one for different areas).

Enter the integral by typing **Integral((Integral(sin(x+y)** [space] **d** [space] **x)** [tab] **0** [tab] **y)*d*y)** [tab] **0** [tab] **1**. Clarify to remove the unnecessary parentheses.

$$\square \int_0^1 \left(\int_0^y \sin[x+y]dx \right) dy$$

The Collapse command of the Notebook menu was applied to the left-hand side of this equation.

First, try to Simplify, and notice that Theorist does not integrate this symbolically.

$$\triangle \cdots\cdots = \Big[\int (dy) \Big(\Big[\int \sin[x+y]dx \Big]_{x=0}^{x=y} \Big) \Big]_{y=0}^{y=1} \qquad \textit{Simplify}$$

The integrals with limits turn into integrals enclosed in Evaluate At operators, but the integrals themselves do not get processed. The inner integral is not evaluated, because it contains the variable, *y*, with an uncertain relationship to the variable of integration, *x*. As far as Theorist knows, *y* could be equal to *x* or some function of *x*, or vice versa. Without knowing this relationship, it is impossible to find an answer.

There is one important thing to do when doing multiple integrals, and indeed when doing most work in two dimensions. You must use an independence declaration. Choose **Independence Decl.** from the **Insert** submenu of the **Notebook** menu. Type **x,y** in the question mark. (The parentheses appear automatically.)

The variables (x, y) are independent of [each other ▾].

Now, one Simplify command gives the exact value of the integral:

$$\triangle \int_0^1 \left(\int_0^y \sin[x+y]\, dx \right) dy = -\frac{1}{2}\sin(2) + \sin(1) \quad \textit{Simplify}$$

Calculating Multiple Integrals

You can also evaluate multiple integrals numerically with the Calculate command. Select the original integral and click [calculator icon] (or choose **Calculate** from the **Manipulate** menu).

$$\triangle \int_0^1 \left(\int_0^y \sin[x+y]\, dx \right) dy = 0.38682 \quad \textit{Calculate}$$

See the Reference Manual's Tips and Techniques section for information on evaluating multiple integrals numerically.

Integrand Approximation

As an example of using an approximation for the integrand, let's modify the previous integral to be more difficult. First, delete the conclusions in the notebook. Select sin[x + y] and click on the functions palette button sin(x).

$$\square \int_0^1 \left(\int_0^y \sin[\sin(x+y)]\, dx \right) dy$$

This expression does not simplify because sin(sin[x]) has no symbolic integral in terms of familiar functions. We could try to split it up using trigonometric identities for addition, but that leads nowhere. (Try it.) Instead, we use a Taylor series.

Taylor Series Approximation

The critical thing about taking a multidimensional Taylor Series is to make the variables independent. If you don't, all of the derivatives with respect to x have trouble with the y's, and things do not simplify as they should. Fortunately, we already have the correct independence declaration lying around; for our purposes, x and y are independent.

Select the integrand (without the dx) and choose [Taylor series icon] from the [hammer icon] pop-up (or choose **Manipulate▸Other▸Taylor Series**). At the dialog, make sure x is selected in the left scroll box and press [return]. (Three terms is fine for now.) When that is completed, keep the selection and choose **Taylor Series** again. This time, select y in the dialog's left scroll box. You end up with an integral of a polynomial.

$$\triangle\ \blacksquare = \int_0^1 \left(\int_0^y \left[-\frac{1}{3}x^3 + x + x^2y^3 - \frac{1}{3}y^3 + x^3y^2 - xy^2 - x^2y + y \right] dx \right) dy \qquad \textit{Taylor Series}$$

If you Expand the whole integral, it reduces to a simple answer.

$$\triangle \int_0^1 \left(\int_0^y \sin[\sin\{x + y\}]\,dx \right) dy = \frac{1}{3} \qquad \textit{Expand}$$

How close is this? Let Theorist approximate the integral by using the Calculate command on the original integral. (It does not take very long on a fast computer.)

$$\triangle \int_0^1 \left(\int_0^y \sin[\sin\{x + y\}]\,dx \right) dy = 0.34165 \qquad \textit{Calculate}$$

This answer is good to about five digits, showing how poor the Taylor series is, especially since the Taylor series took more work and more time. In fact, the numerical integration algorithm is actually very good and fast for smooth functions. It works, but is slower for rapidly oscillating functions. Taylor series only really works for smooth functions, in general, the same kind that the numerical integration algorithm works well for.

COMPLICATED INTEGRAL

This tutorial presents a discussion of:

- Techniques for solving non-trivial integrals
- Substituting single variables for more complex expressions
- Using trigonometric Transformation Rules
- Verifying an integral solution by differentiating
- Creating a transformation rule from the solution obtained

This tutorial concerns itself with the following small but difficult integral:

$$\int \sqrt{a + b(\sin[\theta])^2} \cos(\theta) d\theta$$

In the process, we see how to use substitution and transformation rules in integration.

Since this tutorial is long and involved, and you may want to take a break somewhere along the line, save the notebook with the **Save As...** item from the **File** menu. Save the notebook as "Integral" (Macintosh) or "Integral.The" (Windows). Save it often as you work through this tutorial. Saving often decreases the risk of work lost due to system crashes, power failures, and yes, even your own mistakes. Because you may become so absorbed in this derivation you may forget to save, Theorist can help you out with its automatic saving feature. Choose **AutoSave** from the **File** menu. When this item is checked, Theorist periodically saves the notebook.

At any point in this tutorial, if Theorist asks you to declare a name, press [return] to accept the default declaration.

Start by entering the following expression.

$$\square \int \sqrt{a + b(\sin[\theta])^2} \cos(\theta) d\theta$$

You can see by simplifying that it does not solve immediately. But anybody who does frequent integration can see there is a substitution just dying to be done. That cos(θ) looks like the derivative of that sin(θ), so let's try saying x is sin(θ).

$$\square\ x = \sin(\theta)$$

Of course, you need an expression for θ to substitute into your equation, so isolate θ.

$$\triangle\ \theta = \arcsin(x) \quad \textit{Isolate}$$

Most of the rest of this derivation takes place on the right hand side of this equation, with the left integral remaining unchanged.

Now take this last equation and substitute it into your integral,

$$\triangle \int \sqrt{a + b(\sin[\theta])^2}\cos(\theta)\,d\theta = \int \sqrt{bx^2 + a}\,dx \quad \textit{Substitute}$$

Simplify does not crack this.

Whenever you get an integral with a square root like this, you should use some sort of trig or hyperbolic function as a substitution. The idea is to get the $\sqrt{bx^2+a}$ to dissolve into something simple.

Keep the following table handy.

If your integral has this	**use one of these substitutions for *x***
$\sqrt{x^2+1}$	tan z, sinh z, cot z, csch z
$\sqrt{1-x^2}$	sin z, sech z, cos z, tanh z
$\sqrt{x^2-1}$	sec z, cosh z, csc z, coth z

These are chosen because the functions on the right are transformed by the expressions on the left by way of trig or hyperbolic identities. Some of these substitutions work equally well (or equally poorly) in the same situations. Try a trig function first, and if that does not work, try a hyperbolic.

Notice here that you have to decide upon the signs of a and b. For our purposes, let's assume they are both positive.

In this situation, we start with tangent. Unfortunately, the constants a and b complicate the situation. To make $bx^2 + a$ melt like a trig identity, visualize bx^2 being the $(\tan[z])^2$ part and a being the 1 part. Because a is there, we know the first part really needs to be $a(\tan[z])^2$.

Enter the substitution $bx^2 = a(\tan[z])^2$ and Isolate x, then Expand the result.

$$\triangle\ x = \frac{\sqrt{a}\tan(z)}{\sqrt{b}} \quad \textit{Expand}$$

This should make it all dissolve. Substitute it into the integral, then Expand.

$$\triangle\cdots\cdots = \frac{\sqrt{a}\left(\int \sqrt{a[\tan(z)]^2 + a}\,[\sec(z)]^2\,dz\right)}{\sqrt{b}} \qquad \textit{Expand}$$

Select the $(\tan[z])^2$ and do a Transform manipulation.

$$\triangle\cdots\cdots = \frac{\sqrt{a}\left(\int \sqrt{a[(\sec(z))^2 - 1] + a}\,[\sec(z)]^2\,dz\right)}{\sqrt{b}} \qquad \textit{Transform}$$

Expand the fraction twice to reap the fruits of the substitution.

$$\triangle\cdots\cdots = \frac{a\left(\int [\sec(z)]^3\,dz\right)}{\sqrt{b}} \qquad \textit{Expand}$$

That looks ugly. If it were squared instead of cubed, it may crack immediately. This, however, is difficult. Maybe a hyperbolic substitution will work. In the bx^2 equation you entered, change the tan to sinh to make it $bx^2 = a(\sinh[z])^2$, then watch and wait. Theorist takes a moment to automatically remanipulate all of the dependent conclusions, which saves you several steps. However, Theorist can not match the transformation rule so it can not remanipulate the last few conclusions.

Fortunately, Theorist has a fix for exactly this situation. In the last valid conclusion (above the conclusions with the error messages), Transform the $(\sinh[z])^2$, and in the transformation dialog choose the result $(\cosh[z])^2 - 1$. Keep the selection as it is in the new conclusion (any selection in this proposition will work) and Shift-click the conclusion icon of the first ReManipulate error message. What you are doing is making a multiple selection to link the new manipulation to the non-working manipulation. Choose **Divert Cascade** from the **Manipulate** menu and Theorist remanipulates the error conclusions.

$$\triangle \int \sqrt{b(\sin[\theta])^2 + a}\,\cos(\theta)\,d\theta = \frac{\sqrt{a}\left(\int \sqrt{a[(\cosh(z))^2 - 1] + a}\,\cosh[z]\,dz\right)}{\sqrt{b}} \qquad \textit{Transform}$$

$$\triangle \int \sqrt{b(\sin[\theta])^2 + a}\,\cos(\theta)\,d\theta = \frac{\sqrt{a}\left(\int \cosh[z]\sqrt{a[\cosh(z)]^2}\,dz\right)}{\sqrt{b}} \qquad \textit{Expand}$$

$$\triangle \int \sqrt{b(\sin[\theta])^2 + a}\,\cos(\theta)\,d\theta = \frac{a\left(\int [\cosh(z)]^2\,dz\right)}{\sqrt{b}} \qquad \textit{Expand}$$

Select $(\cosh[z])^2$ and Transform it. (Use the $2z$ result when Theorist asks you which one you want.)

$$\triangle \cdots\cdots = \frac{\alpha\left(\int\left[\frac{1}{2}\{\cosh(2z)+1\}\right]dz\right)}{\sqrt{b}} \qquad \textit{Transform}$$

Expand the entire right-hand side of the equation.

$$\triangle \cdots\cdots = \frac{1}{2}\frac{\alpha z}{\sqrt{b}} + \frac{1}{4}\frac{\alpha \sinh(2z)}{\sqrt{b}} \qquad \textit{Expand}$$

The integral is solved but the job is not yet done. We still have two levels of substitution to crawl back through. We start by substituting back to x.

Make arcsinh predefined.

Locate the last equation for x and Isolate z.

$$\triangle\, z = \operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right) \qquad \textit{Isolate}$$

Now, Substitute it into the equation like so:

$$\triangle \cdots\cdots = \frac{1}{2}\frac{\alpha z}{\sqrt{b}} + \frac{1}{4}\frac{\alpha \sinh(2z)}{\sqrt{b}} \qquad \textit{Expand}$$

This produces the following.

$$\triangle \cdots\cdots = \frac{1}{2}\frac{\alpha \operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)}{\sqrt{b}} + \frac{1}{4}\frac{\alpha \sinh\left(2\operatorname{arcsinh}\left[\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right]\right)}{\sqrt{b}} \qquad \textit{Substitute}$$

Before we go any further, let's pound on that sinh of an arcsinh. Select the sinh[2…] as shown below.

$$+\frac{1}{4}\frac{\alpha \sinh\left(2\operatorname{arcsinh}\left[\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right]\right)}{\sqrt{b}}$$

Transform it. In the dialog, click on the result which *does not* contain exponentials (the second one).

$$\triangle\cdots\cdots = \frac{1}{2}\frac{\alpha\,\mathrm{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)}{\sqrt{b}} + \frac{1}{4}\frac{\alpha\left(2\frac{\sqrt{b}\,x\cosh\left[\mathrm{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)\right]}{\sqrt{\alpha}}\right)}{\sqrt{b}} \quad \textit{Transform}$$

Wow. This turned into a sinh multiplied by a cosh, and Auto Simplify annihilated the sinh of the arcsinh and made it x. Using techniques learned previously, we can deal with the cosh of the arcsinh. Enter a new expression:

y is a wildcard variable. Type ?y to enter it.

$$\square\left(\sinh[y]\right)^2$$

Transform it into the following.

$$\triangle\left(\sinh[y]\right)^2 = \left(\cosh[y]\right)^2 - 1 \quad \textit{Transform}$$

Isolate cosh y.

$$\triangle\ \cosh(y) = \sqrt{\left(\sinh[y]\right)^2 + 1} \quad \textit{Isolate}$$

Substitute the above equation into the big equation like so:

$$+\frac{1}{4}\frac{\alpha\left(2\frac{\sqrt{b}\,x\cosh\left[\mathrm{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)\right]}{\sqrt{\alpha}}\right)}{\sqrt{b}}$$

The result is automatically Simplified.

$$\triangle\cdots\cdots = \frac{1}{2}\frac{\alpha\,\mathrm{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)}{\sqrt{b}} + \frac{1}{2}\sqrt{\alpha}\sqrt{\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)^2 + 1}\,x \quad \textit{Substitute}$$

We can simplify this even further. Do an Expand, then Collect everything inside the big square root, then Expand the right-hand side of the resulting equation.

$$\triangle \cdots\cdots = \frac{1}{2}\frac{\alpha \operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)}{\sqrt{b}} + \frac{1}{2}\sqrt{b x^2 + \alpha}\,x \qquad \textit{Expand}$$

Nice and simple. Now we are ready to substitute back to θ. Get the equation $x = \sin\theta$ and Substitute into the big equation like so:

$$\triangle \cdots\cdots = \frac{1}{2}\frac{\alpha \operatorname{arcsinh}\left(\frac{\sqrt{b}\,x}{\sqrt{\alpha}}\right)}{\sqrt{b}} + \frac{1}{2}\sqrt{b x^2 + \alpha}\,x \qquad \textit{Expand}$$

This produces the following.

$$\triangle \cdots\cdots = \frac{1}{2}\frac{\alpha \operatorname{arcsinh}\left(\frac{\sqrt{b}\sin[\theta]}{\sqrt{\alpha}}\right)}{\sqrt{b}} + \frac{1}{2}\sqrt{b(\sin[\theta])^2 + \alpha}\,\sin(\theta) \qquad \textit{Substitute}$$

This is the final answer.

Verifying the Answer

Whenever you do an integral like this, you should check the answer to make sure it is correct. Copy the right hand side of the equation then type [return] **d(** then paste.

$$\square\; d\left(\frac{1}{2}\frac{\alpha \operatorname{arcsinh}\left[\frac{\sqrt{b}\sin\{\theta\}}{\sqrt{\alpha}}\right]}{\sqrt{b}} + \frac{1}{2}\sqrt{b[\sin\{\theta\}]^2 + \alpha}\,\sin[\theta]\right)$$

Simplify it to process the derivative. The right hand side looks like the following.

$$= \frac{1}{2}\frac{\sqrt{\alpha}\cos(\theta)}{\sqrt{\left(\frac{\sqrt{b}\sin[\theta]}{\sqrt{\alpha}}\right)^2 + 1}}\,d\theta + \frac{1}{2}\frac{b\cos(\theta)(\sin[\theta])^2}{\sqrt{b(\sin[\theta])^2 + \alpha}}\,d\theta + \frac{1}{2}\sqrt{b(\sin[\theta])^2 + \alpha}\,\cos(\theta)\,d\theta$$

Do an Expand, then Collect inside the biggest square root, in the denominator of the first term on the right.

$$\triangle d\cdots\cdots = \frac{1}{2}\frac{\sqrt{a}\cos(\theta)}{\sqrt{\frac{b(\sin[\theta])^2+a}{a}}}d\theta+\cdots\cdots \qquad \textit{Collect}$$

Expand the first term on the right side of the equation, and many things cancel out.

$$\triangle d\cdots\cdots = \frac{1}{2}\frac{a\cos(\theta)}{\sqrt{b(\sin[\theta])^2+a}}d\theta+\cdots\cdots \qquad \textit{Expand}$$

Collect the whole right side of the equation.

$$\triangle d\cdots\cdots = \frac{1}{2}\frac{\left(2b[\sin\{\theta\}]^2+2a\right)\cos(\theta)}{\sqrt{b(\sin[\theta])^2+a}}d\theta \qquad \textit{Collect}$$

It looks as though the top and bottom will cancel out some. In the numerator, select $2b[\sin\{\theta\}]^2+2a$ and Collect:

$$\triangle d\cdots\cdots = \frac{1}{2}\frac{\left(2\left[b\{\sin(\theta)\}^2+a\right]\right)\cos(\theta)}{\sqrt{b(\sin[\theta])^2+a}}d\theta \qquad \textit{Collect}$$

Simplify the right side of the equation.

$$\triangle d\cdots\cdots = \sqrt{b(\sin[\theta])^2+a}\,\cos(\theta)d\theta \qquad \textit{Simplify}$$

This was our original integrand. Our answer was right.

Creating a Transformation Rule

If you find yourself having to do the same integrals over and over many times, you could make a rule. Locate the equation where the integral was finally solved.

$$\triangle\cdots\cdots = \frac{1}{2}\frac{a\,\mathrm{arcsinh}\left(\frac{\sqrt{b}\sin[\theta]}{\sqrt{a}}\right)}{\sqrt{b}}+\frac{1}{2}\sqrt{b(\sin[\theta])^2+a}\,\sin(\theta)$$

Click anywhere within the equation. Then, from the **Notebook** menu's **Insert** submenu, select **Transform Rule**, which makes the following rule.

Upon [Transform ▾] transform $\int \sqrt{b(\sin[\theta])^2 + a}\cos(\theta)d\theta$ into

$$\frac{1}{2}\frac{a\,\mathrm{arcsinh}\left(\frac{\sqrt{b}\sin[\theta]}{\sqrt{a}}\right)}{\sqrt{b}} + \frac{1}{2}\sqrt{b(\sin[\theta])^2 + a}\sin(\theta).$$

There are some changes you may want to make to the rule before implementing it. For instance, you may want it to be done upon Simplify instead of upon Transform. Also, you may want to change the θ to a wildcard variable such as 𝕥, so it works for similar equations, instead of just situations where your variable happens to be θ. While you are at it, change the a and b to 𝕒 and 𝕓. Note Theorist allows 𝕒 and 𝕓 to match only constants, whereas the 𝕥 matches variables.

Differential Equations

Solving First Order Differential Equations Symbolically

This tutorial will examine two of the several different classifications of first-order differential equations that can sometimes be solved exactly: separable equations and linear equations. Each class has a particular technique (or trick) that can be used to solve them.

Separable Equation

This type of equation arises, for example, in population modeling.

We'll begin by solving the following simple, separable differential equation:

$$\frac{dN}{dt} = \gamma N$$

Enter the equation by typing **(d*N)/(d*t)** [esc] **='g*N**, or by using the palette. Choose **Clarify** from the **Notebook** menu. Theorist shows a dialog for each undeclared name. For *N* and *t* you can press [return] to make them variables, but use the dialog's pop-up menu to make γ a constant.

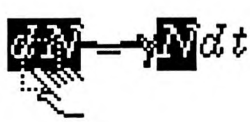
When clarifying Theorist eliminates unnecessary parentheses.

$$\frac{dN}{dt} = \gamma N$$

Now we are ready to start working on the equation. Select *dt* and choose the Move Over tool from the function palette's pop-up (or choose **Move Over** from the **Manipulate** menu's **Other** submenu). Notice how the conclusion generated is a working statement.

$$dN = \gamma N dt \quad \textit{MoveOver}$$

From the expression on the right side of the equal sign, select *N*. Move Over *N*, but this time hold the [⌘] key (Macintosh) or [Ctrl] key (Windows), drag over like so,



...and drop it. This results in:

$$\frac{1}{N} dN = \gamma dt \quad \textit{MoveOver}$$

With the equation selected as above, click or choose **Apply** from the **Manipulate** menu. This has the effect of selecting the expressions on both sides of the equal sign, but not the whole equation with the equal sign:

$$\triangle \; \frac{1}{N} dN = y\,dt$$

Next we want to integrate both sides of the equation. Type **$** for an indefinite integral.

$$\triangle \int \frac{1}{N} dN = \int y\,dt$$

From the **Manipulate▸Preferences** menu, turn on the **Auto Casing** option. This puts Theorist in strict derivation mode, in which it pays close attention to the subtle details and rules of mathematics. We are using it now to generate some special constants, explained shortly hereafter.

Click on the equal sign to select the whole equation, then click (or choose **Simplify** from the **Manipulate** menu).

$$\triangle \; \ln(N) + c_{106} = yt + c_{107} \qquad \textit{Simplify}$$

To solve for N, select it and click (or choose **Isolate** from the **Manipulate** menu) to get the answer:

$$\triangle \; N = e^{yt + c_{108}} \qquad \textit{Isolate}$$

You might have slightly different index numbers like 100 and 101.

The constants c_{106}, c_{107}, and c_{108} were generated by Theorist when you integrated the differential equation. They are arbitrary; any value you choose still works as a solution to the original differential equation. (See a calculus or differential equations book for more details.)

If this were a real-world problem, you could solve for c_{108} from your initial condition.

First-Order Linear Equation

A first-order linear differential equation of the form:

$$\frac{dy}{dx} + y\,P(x) = Q(x)$$

where both $P(x)$ and $Q(x)$ are arbitrary functions of x only, can be solved using an integrating factor,

$$e^{\int P(x)\,dx}$$

The solution to this type of differential equation is given by the following equation:

$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}\,dx + C$$

To demonstrate the use of this method, we will solve the following first-order linear differential equation:

$$\frac{dy}{dx} + 2xy = 4x$$

To enter this equation, type [return] **/d*y** [tab] **d*x** [esc] **+2x*y=4x**.

$$\square \quad \frac{dy}{dx} + 2xy = 4x$$

This is entered primarily to show the equation we want to solve. However, it can also be used later to check that the solution we find is indeed a solution to the given differential equation.

Upon examining the equation, we see that $P(x) = 2x$ and $Q(x) = 4x$. So, let's create the integrating factor. Type [return] **e^$2*x*d*x**. Turn off Auto Casing, which was left on for our previous derivation. Select the expression and click [Simplify icon] (or choose **Simplify** from the **Manipulate** menu).

$$\triangle \quad e^{\int 2x\,dx} = e^{x^2} \qquad \textit{Simplify}$$

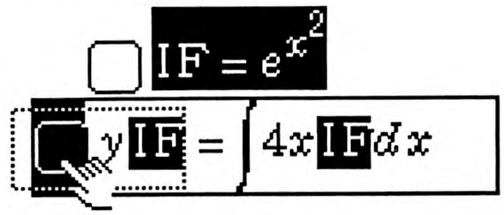
Note that Theorist does not allow you to overwrite a conclusion, but creates a new proposition.

Let's define the result as the integrating factor and call it IF. Select the left side and replace it by typing **IF**. The resulting proposition is:

$$\square \quad \mathrm{IF} = e^{x^2}$$

Clarify the notebook and declare IF as a variable.

Type [return] **y*IF=$4x*IF*d*x** for the solution, then substitute the equation for IF into the new equation, like so:



This produces:

$$\triangle\ e^{x^2} y = 4\left(\int e^{x^2} x\,dx\right) \qquad \textit{Substitute}$$

We are left with the task of evaluating the integral on the right side. Theorist will not evaluate it directly by simplifying, but it can be solved readily using a substitution.

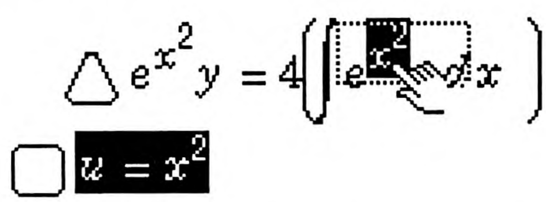
Type [return] **u=x^2**, the equation for the substitution, press [return] and type the differential, **d*u**. Clarify and make *u* a variable. Substitute this first expression into the second.

$$\triangle\ du = 2x\,dx \qquad \textit{Substitute}$$

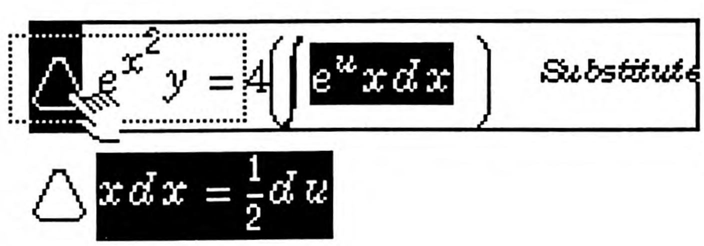
Solve for *x dx* by selecting it and clicking (or choosing **Isolate** from the **Manipulate** menu).

$$\triangle\ x\,dx = \frac{1}{2}du \qquad \textit{Isolate}$$

Turn Auto Casing back on. Select the expression for *u* and substitute it into the integral above, like so:



Select the expression for *x dx* and substitute it into the preceding result, like so:



The result of these operations is:

$$\triangle\ e^{x^2} y = 4\left(\frac{1}{2}e^{u} + c_{110}\right) \qquad \textit{Substitute}$$

With Auto Simplify on, the integral is automatically integrated!

Turn Auto Casing back off. Substitute u back into the result above, then solve the resulting equation for y (with Isolate). Turn Auto Casing back on. Select the entire right side, then click (or choose **Expand** from the **Manipulate** menu).

$$\triangle\ y = c_{111}e^{-x^2} + 2 \qquad \textit{Expand}$$

Select the c_{111} and type **C**, which renames the constant of integration as C in a new proposition.

$$\square\ y = Ce^{-x^2} + 2$$

Finally, we see that this final equation for y is a solution of the given first-order differential equation. It is left to the reader to prove this (just substitute the resulting equation for y into the original differential equation, and expand the resulting left side).

SOLVING SECOND ORDER DIFFERENTIAL EQUATIONS SYMBOLICALLY

Linear differential equations of second (and higher) order, which have constant coefficients, can be solved using Theorist by using the differential operator D. Using the differential operator D (declared as a variable in Theorist) reduces the problem to solving for the roots of a polynomial. The solution depends upon the roots of the polynomial—distinct, repeated, or complex. The examples in this manual should allow you to extrapolate Theorist's symbolic capabilities to other solution techniques, such as variation of parameters and the method of undetermined coefficients.

This example illustrates a procedure for solving the following second-order linear differential equation.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$$

Type **((D^2)-4D+5)y=0**. Clarify the notebook and declare D as a variable.

$$\square\left(D^2 - 4D + 5\right)y = 0$$

Select the polynomial (actually a quadratic) inside the parentheses by dragging or double-clicking on either the plus or minus sign. Click (or choose **Factor** from the **Manipulate** menu).

$$\triangle\left([D-2+i][D-2-i]\right)y = 0 \quad \textit{Factor}$$

Upon examination, we see that the roots are complex ($D = 2 \pm i$), so, according to theory, the solution is given by:

$$y = e^{2x}(C_1 \cos[x] + C_2 \sin[x])$$

To verify this, let's start by typing the equation for the solution: **y=e^2x** esc **(C_1** esc **cos(x)+C_2** esc **sin(x))**. Clarify the notebook, declaring C as a constant.

$$\square\ y = e^{2x}\left(C_1 \cos[x] + C_2 \sin[x]\right)$$

Now enter the differential equation by typing a template, **()^2** esc **y-4()y+5y=0**.

$$\square\ (?)^2 y - 4(?)y + 5y = 0$$

Shift-drag to select each question mark between parentheses. Click $\frac{\partial}{\partial a}$ from the functions palette and type **x** to replace the two question marks in the denominators of the partial differential operators.

$$\square \left(\frac{\partial}{\partial x}\right)^2 y - 4\frac{\partial}{\partial x}y + 5y = 0$$

Exercise

To prove that the solution we found above is a solution, substitute the equation for y into the differential equation above and expand the left-hand side to verify.

BESSEL'S DIFFERENTIAL EQUATION

This tutorial uses some of the rules and declarations in the Finite & Infinite Series notebook to solve Bessel's differential equation using power series (i.e., by finding its Taylor series). The tutorial describes:

- Bessel's differential function
- Creating a power series with the summation operator
- Using an Independence Declaration
- Using a Transformation Rule
- Manipulating an expression in place
- Creating independent case theories

Enter Bessel's differential equation.

Typing hint: You can enter ν (Greek nu) by typing **'n**.

$$\square\left(x^2\left[\frac{\partial}{\partial x}\right]^2+x\frac{\partial}{\partial x}+x^2-\nu^2\right)y=0$$

In this equation, y is a function of x, and ν is a constant, usually an integer. Clarify, making ν a constant.

Start out by assuming a solution in the form of a generic power series for y in terms of x:

Typing hint: You can enter ∞ by typing **'N**.

$$\square\, y=\sum_{k=0}^{\infty} b_k x^k$$

Clarify the notebook and make ∞ predefined. The other names should already be declared. Note that k must be a variable to work in the summation.

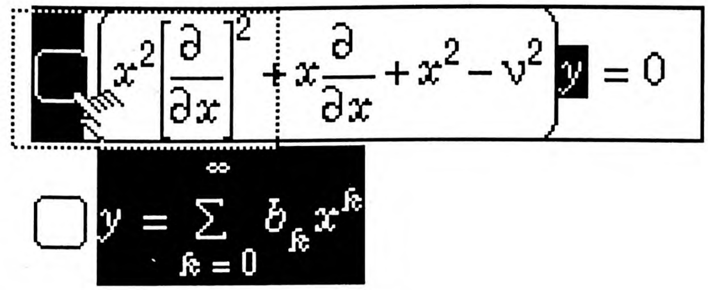
We have to convince Theorist that k will not change in value as x changes in value (as if perhaps k = floor[x]). In addition, we need to specify that b_k will not change in value either; if k can change, then b_k can, too. In order to make this work, we need to specify that k and b_k are independent of x. The easiest way is to declare that they are independent of every other variable.

From Theorist's **Notebook▸Insert** submenu, choose **Independence Decl.** With the question mark selected, type **k,b_k**. Change this declaration's pop-up menu to "all variables".

The variables (k, b_k) are independent of [all variables ▾].

This way, the x's and the derivatives can pass through the Σ.

Take the summation equation and substitute it into the differential equation by dragging (with the ⌘ or Ctrl key) like so:



...to get the following result.

$$\triangle \left(x^2\left[\frac{\partial}{\partial x}\right]^2 + x\frac{\partial}{\partial x} + x^2 - \nu^2\right)\sum_{k=0}^{\infty} b_k x^k = 0 \qquad \textit{Substitute}$$

Expand the result three times.

Part of this long expression is collapsed here for space considerations.

$$\triangle\, x^2 \sum_{k=0}^{\infty} b_k\, k^2 x^{k-2} + \cdots\cdots = 0 \qquad \textit{Expand}$$

This answer is large. We want some of the terms to cancel out, but we have to help Theorist. Situations where there are things multiplied together that do not commute (such as the summations and variables here) can be confusing to Expand and Simplify, which already have their hands full doing other things. To prevent accidents, they conservatively prohibit you from doing things that may not be correct. In such situations, you have to help Theorist along by using Commute. Commute is simpler, because it only has to worry about a few factors at a time, but it more fully understands what commutes with what, and it allows you the full freedom to do what you need to do.

Now, select x^2 from the first term and move it to the right, past the Σ sign, so it is next to the x^{k-2} as shown below.

You may have to drag carefully to get the location you want to highlight.

This produces the following conclusion.

$$\triangle \sum_{k=0}^{\infty} b_k\, k^2 x^2 x^{k-2} + \cdots\cdots = 0 \qquad \textit{Commute}$$

Now we must do Commute three more times, but to avoid the clutter of additional conclusions we will do the Commute manipulation in place so no new conclusions are generated.

To Commute in place, select and drag as before, but hold the ⌘ and option keys (Macintosh) or alt and Ctrl keys (Windows) prior to dragging.

Commute in place each term that starts with an x or an x^2. Move it to the right, past the Σ next to the x^k, x^{k-1}, or x^{k-2}. After the fourth Commute, you end up with:

$$\Delta \sum_{k=0}^{\infty} b_k k^2 x^2 x^{k-2} - \sum_{k=0}^{\infty} b_k k x^2 x^{k-2} + \sum_{k=0}^{\infty} b_k k x x^{k-1} - v^2 \sum_{k=0}^{\infty} b_k x^k + \sum_{k=0}^{\infty} b_k x^2 x^k = 0$$

Simplify the whole equation. The x's unify, and two large terms cancel each other, to yield:

$$\Delta \sum_{k=0}^{\infty} b_k x^{k+2} - v^2 \sum_{k=0}^{\infty} b_k x^k + \sum_{k=0}^{\infty} b_k k^2 x^k = 0 \qquad \textit{Simplify}$$

Since this is true for all x, our aim is to equate coefficients. We want to collect the second two terms together. If you try it, you will notice nothing happens. Collect must be helped along by Commute. When non-commutativity raises its ugly head, Collect can only collect factors that are on the far left or far right.

To help it along, select the v^2 and move it between b_k, and k^2 in the middle summation, as shown below.

$$\Delta \sum_{k=0}^{\infty} b_k x^{k+2} - v^2 \sum_{k=0}^{\infty} b_k x^k + \sum_{k=0}^{\infty} b_k k^2 x^k = 0$$

This gives you:

$$\Delta \sum_{k=0}^{\infty} b_k x^{k+2} - \sum_{k=0}^{\infty} b_k v^2 x^k + \sum_{k=0}^{\infty} b_k k^2 x^k = 0 \qquad \textit{Commute}$$

Next, select the last two summation terms and click (or choose **Collect** from the **Manipulate** menu).

$$\Delta \sum_{k=0}^{\infty} b_k x^{k+2} + \sum_{k=0}^{\infty} \left(k^2 - v^2\right) b_k x^k = 0 \qquad \textit{Collect}$$

Change the x^{k+2} in the first term to x^k because we want the x's to line up. To shift the whole summation over by two, we will make use of a transformation rule in a notebook included with Theorist.

Be careful to get the last rule in the bundle, the one with $s = s - 1$ in the parentheses.

Open the Finite & Infinite Series notebook. (On the Macintosh, it is in the Mathematics folder. On Windows, the notebook is named FinInf.The and is in the Math directory.) Expose the "Joining, Unraveling and Shifting" bundle and select the following transformation rule.

Upon [Transform ▼] transform $\sum_{s=a}^{b} P$ into $\sum_{s=a+1}^{b+1} \left(s = s-1 \left[P \right] \right)$.

Copy this rule to the clipboard and go back to your working notebook and paste it in. (If you like, you may move it under Declarations to keep it out of the way.)

Select the whole first term in our new assumption.

$$\square \quad \sum_{k=0}^{\infty} b_k x^k + \sum_{k=0}^{\infty} \left(k^2 - v^2\right) b_k x^k = 0$$

Click (or choose **Transform** from the **Manipulate** menu). Once you get an answer, keep the selection (which is the first term) and Transform again to move it over one more step.

$$\triangle \quad \sum_{k=2}^{\infty} b_{k-2} x^{k-2} + \sum_{k=0}^{\infty} \left(k^2 - v^2\right) b_k x^k = 0 \qquad \textit{Transform}$$

Instead of typing these from scratch, you can copy and paste pieces from the last conclusion.

Next, generate the recurrence relations, one for each value of *k*, by creating the following new assumptions.

$$\square \quad \left(k^2 - v^2\right) b_k = 0$$

$$\square \quad b_{k-2} + \left(k^2 - v^2\right) b_k = 0$$

The first is only used for k values of 0 and 1. Enter new assumptions for $k = 0$ and $k = 1$. Substitute each of them, one at a time, into the equation $(k^2 - v^2)b_k = 0$. This generates the equations for 0 and 1.

$$\triangle \quad -v^2 b_0 = 0 \qquad \textit{Substitute}$$

$$\triangle \quad \left(-v^2 + 1\right) b_1 = 0 \qquad \textit{Substitute}$$

Select b_k from the other new assumption and click (or choose **Isolate** from the **Manipulate** menu).

$$b_k = -\frac{b_{k-2}}{k^2 - \nu^2} \quad \textit{Isolate}$$

The three conclusions we have here are all we need to generate all of the coefficients for our polynomial.

Remember, type **'n** for the Greek letter nu, ν, not the letter v.

Let's first examine the case where ν = 0. Click some white space in the notebook to de-select, then choose **Notebook▸Insert▸Case Theory**. Enter ν = 0 in its blank assumption. Substitute this into each of our three magic conclusions from the last step.

$$\nu = 0$$

$$0 = 0 \quad \textit{Substitute}$$

$$b_1 = 0 \quad \textit{Substitute}$$

$$b_k = -\frac{b_{k-2}}{k^2} \quad \textit{Substitute}$$

You will notice that we get a firm message that b_1 is zero, but nothing about b_0, which is arbitrary. Enter $b_0 = 1$ in a new assumption within this case theory. Also enter $k = 2$ in a new assumption within this case theory.

Substitute $k = 2$ into our equation for b_k. Enter a new assumption, $k = 4$, and repeat, then do it again for $k = 6$. (We are skipping 3 and 5 because they end up as zero.)

$$b_2 = -\frac{1}{4} b_0 \quad \textit{Substitute}$$

$$b_4 = -\frac{1}{16} b_2 \quad \textit{Substitute}$$

$$b_6 = -\frac{1}{36} b_4 \quad \textit{Substitute}$$

Now, substitute these equations until you have numerical values for all of them.

$$\triangle\; b_2 = -\frac{1}{4} b_0 \quad \textit{Substitute}$$

$$\triangle\; b_2 = -\frac{1}{4} \quad \textit{Substitute}$$

$$\triangle\; b_4 = -\frac{1}{16} b_2 \quad \textit{Substitute}$$

$$\triangle\; b_4 = \frac{1}{64} \quad \textit{Substitute}$$

$$\triangle\; b_6 = -\frac{1}{36} b_4 \quad \textit{Substitute}$$

$$\triangle\; b_6 = -\frac{1}{2304} \quad \textit{Substitute}$$

$$\square\; b_0 = 1$$

A check with *Handbook of Mathematical Functions* (Abramowitz and Stegun, Dover Publications) tells us that we have found the coefficients for $J_0(x)$, the most well-known of the Bessel functions. If we were to substitute in $\nu = 1$, we could get the coefficients for $J_1(x)$, which only has coefficients for odd values of k. Other values for ν would yield the full set.

It turns out that there is another solution to this equation that can not be modeled as a power series because it has a singularity at $x = 0$. These functions are the Bessel functions of the second kind, known collectively as $Y_n(x)$. Together the *J's* and the *Y's* and their linear combinations constitute a full space of solutions, when ν is an integer.

Solving First Order Differential Equations Numerically

As the use and power of digital computers has increased, the numerical solution of differential equations has become common practice in many fields. Solving differential equations using numerical techniques allows many equations to be solved (most notably non-linear) that can not be solved using traditional methods (i.e., integration, algebraic, Laplace Transforms, etc.)

This and subsequent chapters describe how Theorist can be used to numerically solve differential equations. The solution of a first order differential equation is stored in a table. The data in this table can be examined and plotted like other Theorist tables.

Let's use Theorist to find the solution to the following differential equation,

$$\frac{dx}{dt} = -\frac{13}{3}x$$

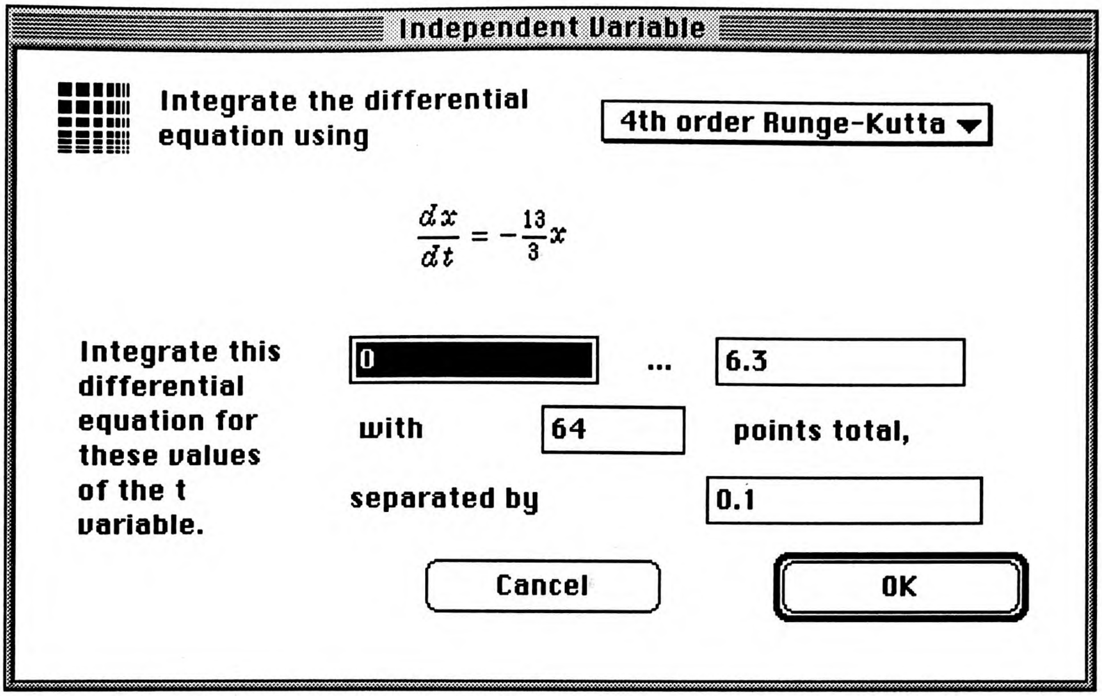
...with the initial conditions $t = 0$ and $x = 1$.

Type **(d*x)/d*t** [esc] **=-13/3** [esc] **x** to make the differential equation. Choose **Clarify** from the **Notebook** menu and declare t as a variable.

$$\square \; \frac{dx}{dt} = -\frac{13}{3}x$$

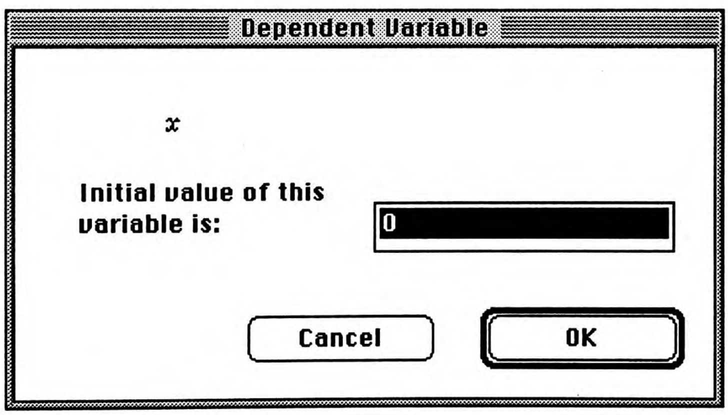
Note you can use both differentials and partials for writing the ordinary differential equations in Theorist.

Select the differential equation, then choose **Manipulate▸Table▸Integrate Differential Equation...**. The following dialog appears.

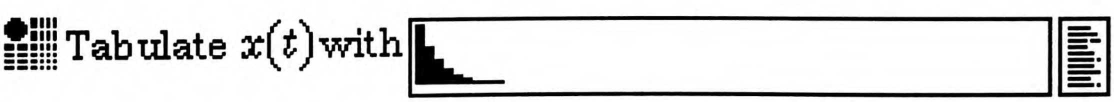


Click **OK**.

Next, you are presented with a dialog asking for the initial value of x.



Enter 1 for the initial value for x and click **OK**. Theorist generates a table proposition below the equation.



Change the name of the resulting table from $x(t)$ to Xr(t), then clarify the notebook and press [return] to make Xr a user-defined variable.

Repeat the preceding steps, but set the big dialog's pop-up menu to Euler's method, and after integrating change the name of the resulting table to Xe(t) and Clarify to make it a variable.

So far, your notebook looks like this:

$$\frac{dx}{dt} = -\frac{13}{3}x$$

Tabulate Xr(t) with

Tabulate Xe(t) with

Type [return] **x=e^-13/3** [esc] **t** for the solution to the differential equation. (Or, derive it from the original equation and boundary conditions.)

Trust us; this is correct.

$$x = e^{-\frac{13}{3}t}$$

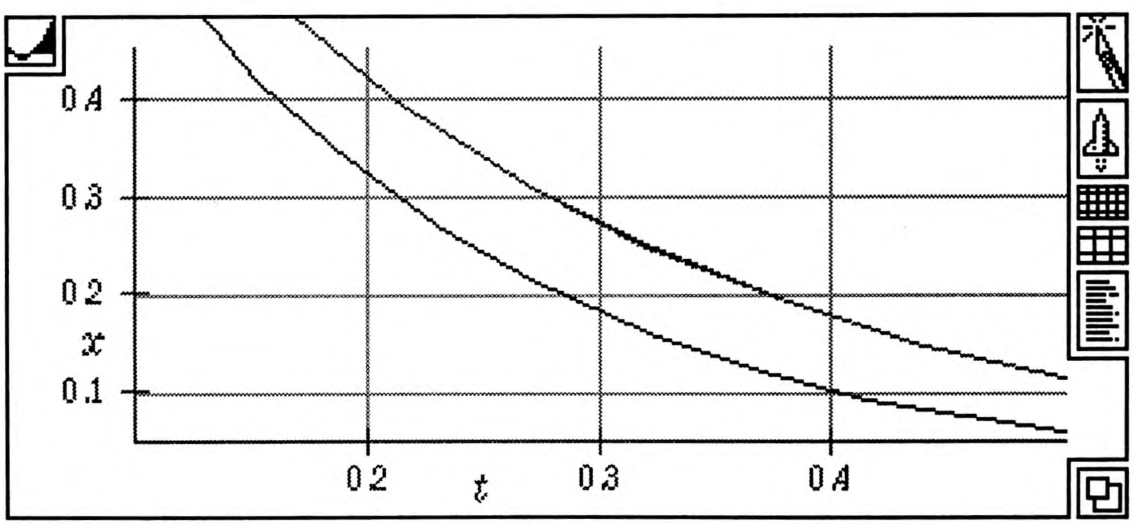
With the selection anywhere within this equation, click [graph icon] or choose **Graph▸y = f(x)▸Linear**. Next we add the Euler (Xe) and Runge-Kutta (Xr) solutions to the graph. Select the table for Xr, then choose **Add Line Plot** from the **Graph▸Additional** submenu. (Accept the defaults in the axis variables dialog.) Repeat for the Xe Euler solution. To examine the results more closely, open the graph details and change the graph bounds as follows.

0.1… 0.5= left…right

0.05 … 0.45 = bottom…top

The graph appears as follows.

Notice the original x black line and added Xr red line appear so close together, they almost can not be distinguished as separate plots.



Compare solutions using both Euler and Runge-Kutta. Note the difference between the two methods of solution. The Runge-Kutta method should be used in almost all situations, whereas the Euler method is included only for educational purposes.

Both algorithms inch along by estimating dx/dt over an increment and then advancing x and t. Euler simply evaluates the derivative at the beginning of the increment, ignoring the fact it will change from one end of the increment to the other. It is a lopsided method which achieves poor results.

Runge-Kutta, however, evaluates the derivative four times (at the start, at the end, and twice in the middle under slightly different conditions) and uses a weighted average of the four. The results are far more accurate than Euler's method, as you can see. But, for the same situation, it uses three to four times more processing time. You might be tempted to think Euler is useful if you need fast results; this is not true.

You can speed up the solver by choosing fewer points in the dialog, and you can increase the accuracy by choosing more points. Let's see if we can improve Euler to the quality level of Runge-Kutta.

Select the differential equation and choose **Manipulate▸Table▸Integrate Differential Equation...** again. In the big dialog, set the pop-up menu to **Euler's method** and change the 64 to 512. (This makes the steps eight times smaller and causes Theorist to take eight times as long to solve the ordinary differential equation—more than double the time it took with the Runge-Kutta method.) In the small dialog, use 1 again for the initial value. Change the resulting table's name to Xeb(t) and Clarify to declare Xeb as a variable. With the table selected, add a line plot and notice the new yellow plot on the graph. This most recent use of Euler's took twice as long as Runge-Kutta but still produced poorer results.

Exercise

Try using larger step sizes on the Runge-Kutta algorithm until its results are as poor as the best Euler's method.

Solving Second Order Differential Equations Numerically

Let's use Theorist to find a numerical solution to the following second order differential equation,

$$\frac{d^2y}{dx} - 4\frac{dy}{dx} + 20y = 0$$

...with the following initial conditions.

$$y(0) = 0, \frac{dy}{dx} = 0.043214$$

To solve this differential equation, it is necessary to form a system of first order differential equations. The original differential equation may be decomposed into the following:

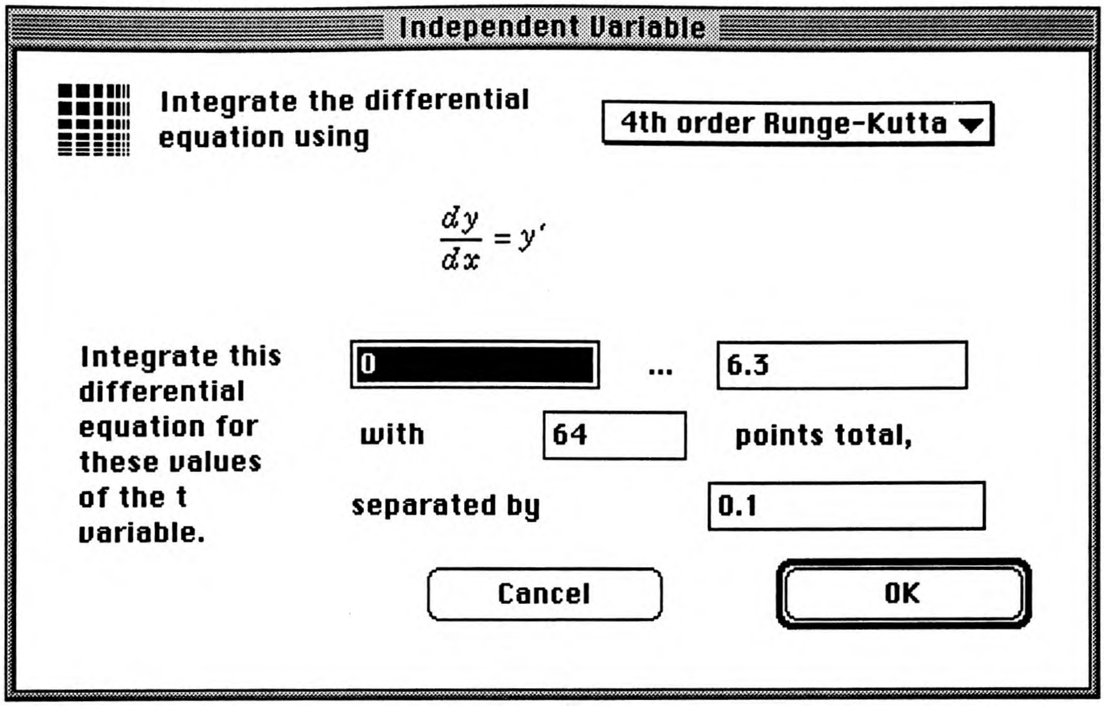
$$\frac{dy}{dx} = y'$$
$$\frac{dy'}{dx} = 4y' - 20y$$

Enter these equations so you have the following propositions.

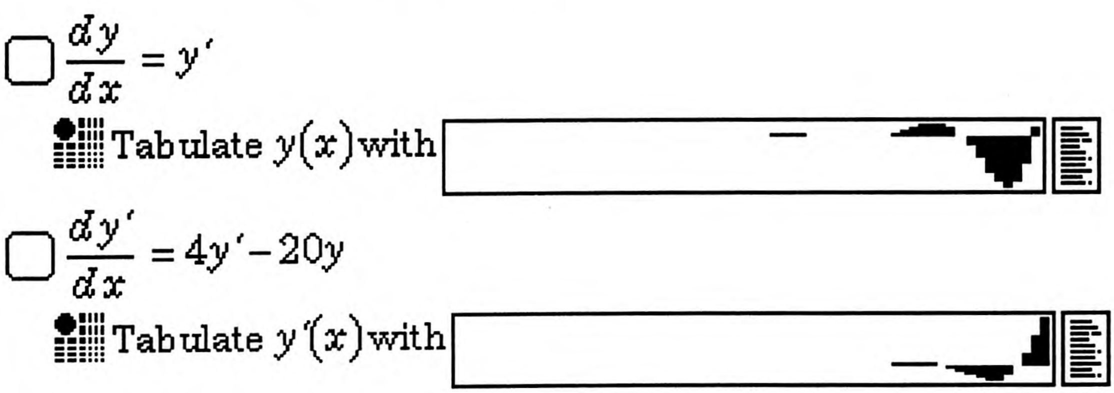
$$\square \; \frac{dy}{dx} = y'$$
$$\square \; \frac{dy'}{dx} = 4y' - 20y$$

Clarify the notebook and let y' be a variable.

Select the first proposition, then shift-click the second so they are both selected. Choose **Manipulate▸Table▸Integrate Differential Equation...**. The big dialog appears; make sure it uses the Runge-Kutta method with 64 points,



...then click **OK**. In the dialog that follows for *y*, use 0 as the initial condition, then in the dialog for *y*', use 1. After a few seconds (depending upon the speed of the machine you are using), two tables appear under the original propositions.



At this point there are many interesting ways to graph the data. One good way is to leave the selection exactly as it is and choose **Graph▸Other▸x = f(t), y = g(t) Parametric**. (Accept the defaults in the axis variables dialog.) This graphs *y* versus *y*' and makes a spiral which starts small and grows larger, as the table displays suggest.

SOLVING HIGHER ORDER DIFFERENTIAL EQUATIONS NUMERICALLY

To solve higher order differential equations, it is necessary to write the *n*th order ordinary differential equation as a system of *n* first order differential equations,

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2, \cdots, y_n)$$
$$\frac{dy_2}{dt} = f_2(t, y_1, y_2, \cdots, y_n)$$
$$\cdots\cdots\cdots$$
$$\frac{dy_n}{dt} = f_n(t, y_1, y_2, \cdots, y_n)$$

...satisfying the following initial conditions.

$$y_1(t_0) = a_1, y_2(t_0) = a_2, \cdots, y_n(t_0) = a_n$$

Some care must be exercised in writing an *n*th order ordinary differential equation as a system of *n* first order differential equations.

As an example of a more complicated system of first order differential equations, let us examine the Lorenz equations which give rise to the Lorenz attractor.

$$x' = -\sigma x + \sigma y$$
$$y' = Rx - y - xz$$
$$z' = -Bz + xy$$

In the following example, we use the values $\sigma = 10$, B = 8/3, and R = 28 for the parameters in the preceding differential equations, subject to the initial conditions $x = 0, y = 1$, and $z = 0$.

Memory Usage Caution

This tutorial uses a lot of memory. To avoid problems, Macintosh users should allocate more memory to the Theorist application.

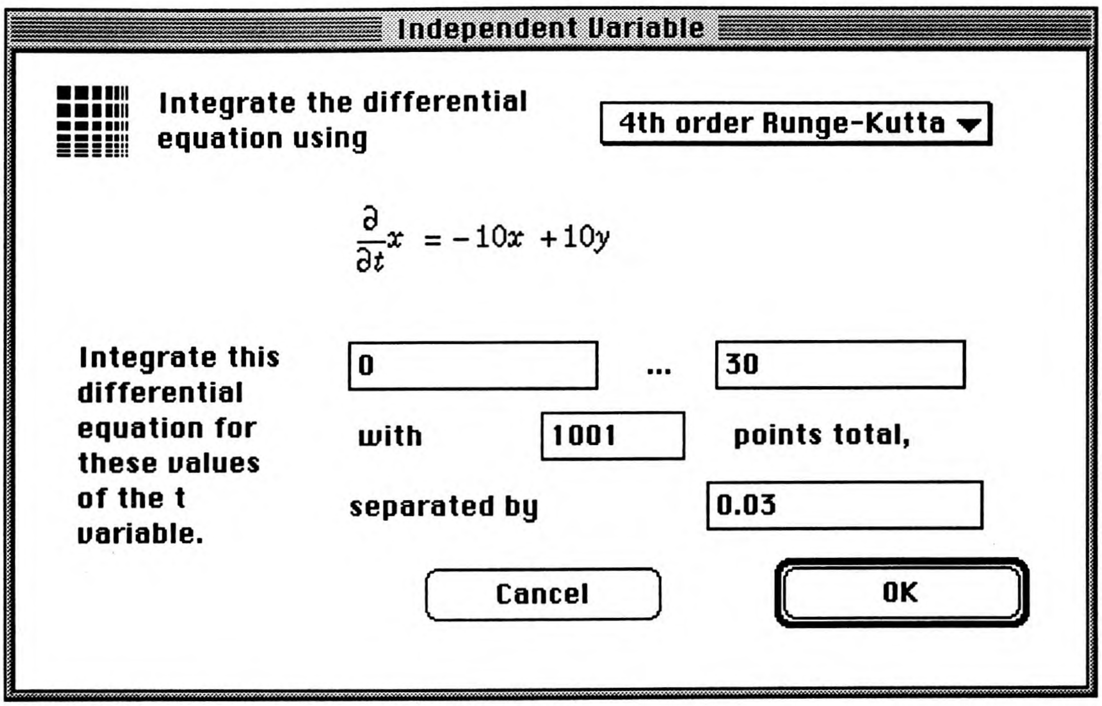
Create a new notebook and enter the Lorenz equations as shown below.

$$\frac{\partial}{\partial t}x = -10x + 10y$$

$$\frac{\partial}{\partial t}y = 28x - y - xz$$

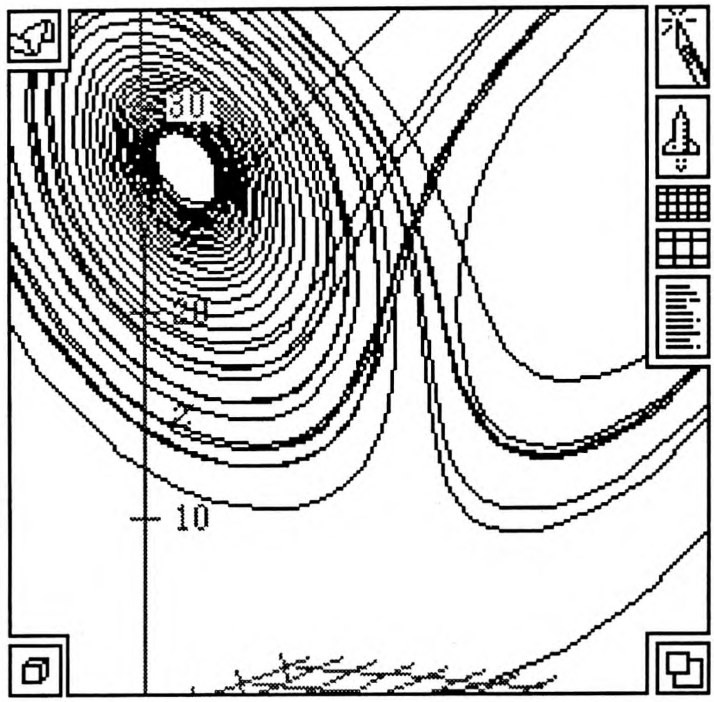
$$\frac{\partial}{\partial t}z = xy - \frac{8}{3}z$$

Clarify and declare t as a variable. Select all three differential equations, then choose **Manipulate▸Table▸Integrate Differential Equation…**. In the big dialog, change the range of values and points as shown below.



In the following dialogs for initial values, use 0 for x, 1 for y, and 0 for z. After integrating, Theorist makes a table below each of the three original differential equations.

To draw a three-dimensional parametric graph, make sure each table is selected and choose **Graph▸Other▸x = f(t), y = g(t), z = h(t) Space Curve**. (Accept the defaults in the axis variables dialog.)



Exercise

For entertainment, change the final time of the system. Change the initial conditions. Change R. (R = 100.5 is a stable periodic solution.)

Matrices and Linear Algebra

BASIC MATRIX OPERATIONS

This tutorial shows:

- Matrix addition
- Matrix multiplication
- Operations between matrices and scalars
- Symbolic and numeric operations

Numeric Multiplication

To enter a matrix in Theorist, use commas to separate elements and semicolons to separate rows. Make a new assumption and type **(1,2;3,4)*(5,6;7,8)**. This gives you:

$$\square \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Now, select both matrices together and click (or choose **Calculate** from the **Manipulate** menu).

$$\triangle \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Calculate

Symbolic Operations

Theorist can also multiply symbolically. Change the 2 in the first matrix to an x. Then select both matrices again and this time click (or choose **Expand** from the **Manipulate** menu).

$$\triangle \begin{pmatrix} 1 & x \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 7x+5 & 8x+6 \\ 43 & 50 \end{pmatrix}$$

Expand

Theorist considers a scalar (a non-matrix, single number) the same as a diagonal matrix. Type [return] **(1,2,3;4,5,6;7,8,9)+x**.

$$\square \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + x$$

Select the expression (by double-clicking on the plus sign), then Expand it (Simplify is not strong enough to do this):

$$\triangle \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + x = \begin{pmatrix} x+1 & 2 & 3 \\ 4 & x+5 & 6 \\ 7 & 8 & x+9 \end{pmatrix} \qquad \textit{Expand}$$

It's as if the x turned into x times the identity matrix. You do not have to make up an identity matrix; Theorist automatically helps out in this situation.

Matrix Inverse

The (multiplicative) inverse of a matrix can be found in several ways. You could use division, but in this case we will follow the typical notation of raising the matrix to the –1 power. Note that the matrix operations can be applied directly to the matrix itself or, as shown here, to a symbolic name which represents the matrix, e.g., *A*, *B*, etc.

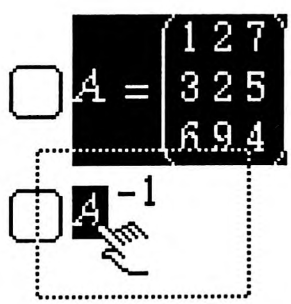
Let's begin by typing **A=(1,2,7;3,2,5;6,9,4)** to create this named matrix:

$$A = \begin{pmatrix} 1 & 2 & 7 \\ 3 & 2 & 5 \\ 6 & 9 & 4 \end{pmatrix}$$

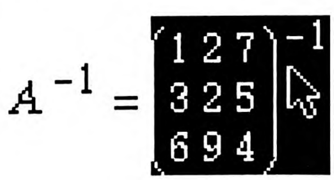
Clarify the notebook and declare *A* to be an M-Linear Operator, the category in Theorist which includes matrices (see the Theorist Reference Manual for more details):

Recall that ^ is the symbol for exponentiation.

Now make another proposition for the inverse of *A*, by typing [return] **A^-1**. Drag the first equation, the one which defines *A*, into the expression for its inverse, like so:



After substituting, select the right-hand side (you can click slightly below the –1 as shown by the cursor below).



Click (or choose **Expand** from the **Manipulate** menu).

$$A^{-1} = \frac{1}{104}\begin{pmatrix} -37 & 55 & -4 \\ 18 & -38 & 16 \\ 15 & 3 & -4 \end{pmatrix} \quad \textit{Expand}$$

Now Expand once more to continue the multiplication process.

$$\triangle A^{-1} = \begin{pmatrix} -\frac{37}{104} & \frac{55}{104} & -\frac{1}{26} \\ \frac{9}{52} & -\frac{19}{52} & \frac{2}{13} \\ \frac{15}{104} & \frac{3}{104} & -\frac{1}{26} \end{pmatrix}$$ *Expand*

Numeric Solution

Calculate can also be used when computing the inverse of a matrix. Generally, Expand is used for operations involving symbolic matrices with variables or irrational numbers, and Calculate is used for numeric operations. To illustrate this, select the definition for the inverse of A as shown below.

$$\square A^{-1}$$

Now click (or choose **Calculate** from the **Manipulate** menu).

$$\triangle A^{-1} = \begin{pmatrix} -0.35577 & 0.52885 & -0.038462 \\ 0.17308 & -0.36538 & 0.15385 \\ 0.14423 & 0.028846 & -0.038462 \end{pmatrix}$$ *Calculate*

Exercise

Try putting a variable x into one of the elements of A and then find its inverse.

General Linear Systems

In Theorist, as in mathematics generally, there is rarely only a single method for solving a problem. The methods for simultaneous linear equations described previously in the Algebra chapter are fine if you only have a few equations, but they can get complicated and tedious if you have larger systems with more variables. Consider this system of equations:

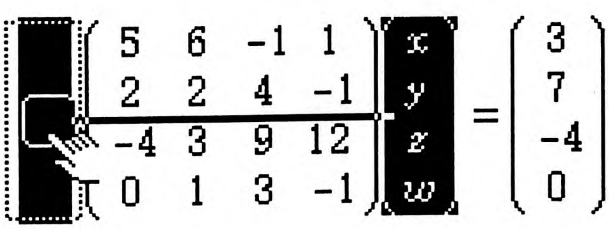
$$\begin{aligned} 5x + 6y - z + w &= 3 \\ 2x + 2y + 4z - w &= 7 \\ -4x + 3y + 9z + 12w &= -4 \\ y + 3z - w &= 0 \end{aligned}$$

The easy way to solve this with Theorist is to express the system as a matrix equation. The solution is found by taking the inverse of the coefficients of the variables. As an application of matrix inversion, this method only works with systems that have the same number of equations as unknowns.

In a new notebook, type **(5,6,-1,1;2,2,4,-1;-4,3,9,12;0,1,3,-1)(x;y;z;w) =(3;7;-4;0)**. Clarify the notebook and define w as a variable, which leaves you with the following.

$$\Box \begin{pmatrix} 5 & 6 & -1 & 1 \\ 2 & 2 & 4 & -1 \\ -4 & 3 & 9 & 12 \\ 0 & 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ -4 \\ 0 \end{pmatrix}$$

Now, simply select and Isolate to solve for the vector with the variables. (You can choose the **Isolate** menu command or palette button or keystroke, or drag to the icon as shown below.)



Note the matrix inverse in the resulting equation:

$$\triangle \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 & 6 & -1 & 1 \\ 2 & 2 & 4 & -1 \\ -4 & 3 & 9 & 12 \\ 0 & 1 & 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 7 \\ -4 \\ 0 \end{pmatrix} \qquad \textit{Isolate}$$

Then, you may Expand twice, for symbolic algebra answers.

$$\triangle \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{35}{8} \\ -\frac{123}{40} \\ \frac{53}{40} \\ \frac{9}{10} \end{pmatrix} \qquad \textit{Expand}$$

Or you may Calculate, for numeric answers. As you might expect, Expand is usually slower than Calculate.

$$\triangle \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4.375 \\ -3.075 \\ 1.325 \\ 0.9 \end{pmatrix} \qquad \textit{Calculate}$$

DETERMINANTS

You can easily evaluate a determinant using Theorist, as this tutorial demonstrates.

Start by turning off Auto Simplify by choosing **Auto Simplify** from the **Manipulate** menu's **Preferences** submenu.

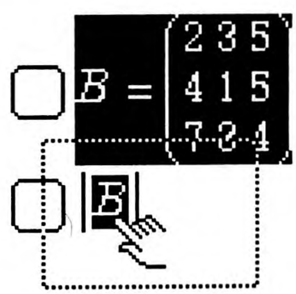
Type **B=(2,3,5;4,1,5;7,3,4)** [return] **|B**. (Theorist provides the closing absolute bar.)

The | character is [shift][\] (backslash) on most keyboards.

$$B = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 5 \\ 7 & 3 & 4 \end{pmatrix}$$

$$|B|$$

Clarify and declare *B* as a M-Linear Operator. Drag the equation defining *B* into the expression for the determinant of *B,* like so:



This produces the following.

$$|B| = \left| \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 5 \\ 7 & 3 & 4 \end{pmatrix} \right| \quad \textit{Substitute}$$

Select the matrix (with its bars) and Expand or Calculate.

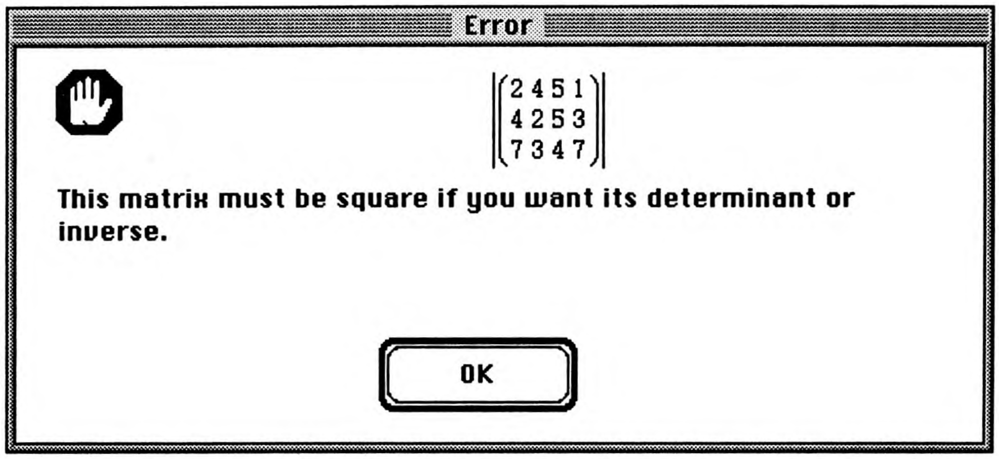
$$|B| = 2(1 \cdot 4 - 3 \cdot 5) - 4(3 \cdot 4 - 3 \cdot 5) + 7(3 \cdot 5 - 1 \cdot 5) \quad \textit{Expand}$$

$$|B| = 60 \quad \textit{Calculate}$$

If we had not turned off Auto Simplify, the substitution would have been automatically simplified to $|B| = 60$.

Note

Remember that determinants can only be calculated for square matrices. If the matrix you select is not square, you will get a warning message from Theorist.



SIMPLE LINEAR SYSTEMS

Whenever you have multiple equations with multiple unknowns, the general strategy is to eliminate variables one by one until you have just one equation in one unknown. Then you solve it. To eliminate a variable, take any equation, solve for the variable you want to eliminate in terms of the others, then substitute it into all the rest of the equations.

Enter these two equations:

$$2 = 3x+4y$$
$$5 = 6x-7y$$

This is a traditional "two linear equations in two unknowns" problem. This tutorial shows you how to solve and manipulate equations with Theorist by walking you through this set of equations.

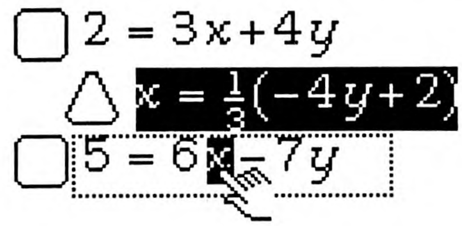
First, select the *x* in the first equation, and drag it as described in the Manipulating Equations tutorial to isolate it.

$$2 = 3x+4y$$
$$x = \frac{1}{3}(-4y+2)$$
$$5 = 6x-7y$$

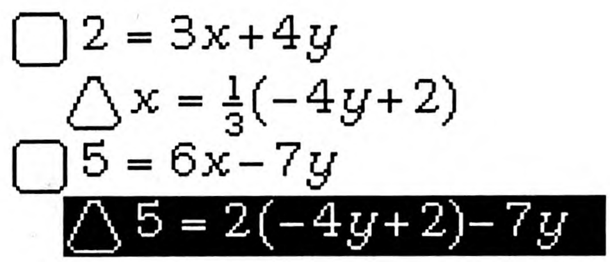
The next step of the derivation is to take the first equation (now rearranged as a conclusion) and plug it into the second equation. Make sure the equation for *x* is selected; if it isn't, click on its equal sign to select it. Press and hold the ⌘ key (Macintosh) or Ctrl key (Windows).

$$2 = 3x+4y$$
$$x = \frac{1}{3}(-4y+2)$$
$$5 = 6x-7y$$

Drag it over the x in the bottom proposition, like so:

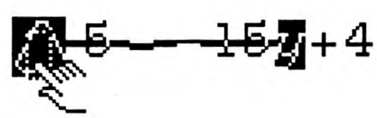


When you end the drag by releasing the mouse button, the substitution is performed.

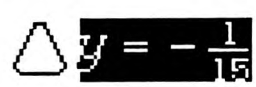


With the last conclusion selected, click (or choose **Expand** from the **Manipulate** menu) to combine the two y's:

Now that you have one occurrence of y, you can solve for it by using Isolate, but this time do it by hand. First select the y and then hold down the ⌘ key (Macintosh) or Ctrl key (Windows) as you drag the y to the left, until it is over the conclusion icon, and release the mouse button:

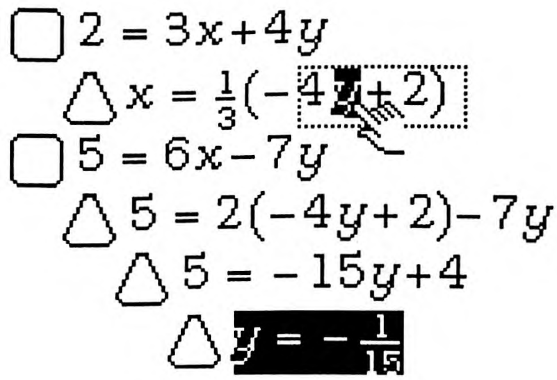


This yields your answer for y:

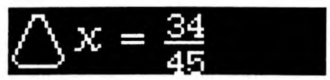


Substitute your answer into the second equation from the top to get an equation for x.

Your answer is already selected, so hold down ⌘ or Ctrl, and drag it to the location shown below.



This gives you:



Exercise Substitute these answers for x and y into the original equation. Prove that it works.

Exercise Use multiple selections to make your work easier. Redo the last exercise by first selecting both the equations for x and for y, and by dragging them over the proposition icon (*not* the variables themselves) of each target equation.

Exercise Enter the following two equations and solve for x and y.

$$a = 3x + by$$
$$c = px + 9y$$

The methods described previously are fine if you only have a few equations, but can get tedious if you have many linear equations. Consider the following system of equations.

$$\begin{aligned} 5x + 6y - z + w &= 3 \\ 2x + 2y + 4z - w &= 7 \\ -4x + 3y + 9z + 12w &= -4 \\ y + 3z - w &= 0 \end{aligned}$$

Solving this would involve moving around several equations and plugging everything into everything and would be really tedious. The easy way to do it is to make it a matrix equation as we did previously in this section's General Linear Systems tutorial.

EIGENVALUES

This tutorial describes how to work with basic Hilbert matrices and determine their eigenvalues and eigenvectors. The tutorial provides a discussion of:

- Creating and manipulating matrices and vectors
- Finding eigenvalues and eigenvectors
- Using the Auto Casing option to see all solutions to a manipulation

Theory Digression

Do not enter any expressions yet.

The eigenvalues and eigenvectors are the solutions of the equation $Mv = \lambda v$ where v is a vector, M is a matrix, and λ is a scalar. Given M, we will first find the eigenvalues λ and then find the eigenvector v which accompanies each eigenvalue λ. To find λ, we must solve $|M - \lambda I| = 0$. In Theorist, you do not use the I identity matrix because Theorist treats scalars as diagonal matrices.

Given a matrix M, there are certain vectors v which, when transformed by M, remain pointing in the same direction (although the length may change). These are called the eigenvectors (or characteristic vectors) of the matrix. The factor of change in the length is the eigenvalue (characteristic value) of the matrix, corresponding to that eigenvector.

Also notice the length of v is arbitrary. If a given vector v_1 is an eigenvector, then so is $-5v_1$. This fact will become important.

Creating a Matrix

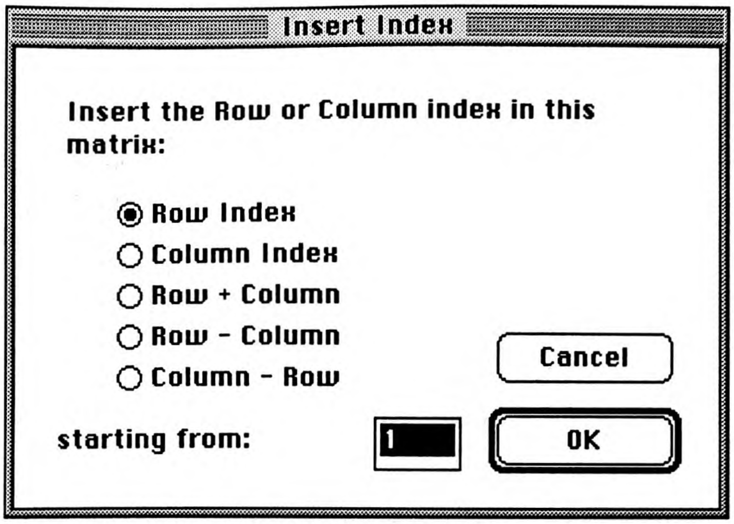
Our first step is to enter a Hilbert matrix. A Hilbert matrix has a regular pattern to its components as follows. Type **M=** and choose the 3×3 matrix button from the functions palette's pop-up matrices sub-palette, which gives you:

$$\square\ M = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}$$

Select the entire matrix by clicking or typing shift esc. All the question marks highlight as a single block. Now click (or choose **Select In** from the **Edit** menu) for nine individual selections.

Next, type **1/1+** and all the selections are entered in unison.

From the **Notebook** menu's **Insert** submenu, choose **Row/Column Number...**, which brings up the Insert Index dialog.



Click the button for Row Index and make the starting number 1. (These are the defaults, so you can just click **OK**.) The row numbers are inserted in the matrix.

Type **+** then choose **Notebook▸Insert▸Row/Column Number...** again. In the dialog, click the button for Column Index, make it start at 1, and click **OK**. You now have a Hilbert matrix.

$$\square\ M = \begin{pmatrix} \frac{1}{1+1+1} & \frac{1}{1+1+2} & \frac{1}{1+1+3} \\ \frac{1}{1+2+1} & \frac{1}{1+2+2} & \frac{1}{1+2+3} \\ \frac{1}{1+3+1} & \frac{1}{1+3+2} & \frac{1}{1+3+3} \end{pmatrix}$$

Select the whole equation by dragging over it or clicking on its equal sign. Hold down the ⌘ *and* option keys (Macintosh) or alt *and* Ctrl keys (Windows) simultaneously as you double click on the equation. After you declare M as an M-Linear op, Theorist gives you:

$$\square\ M = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$$

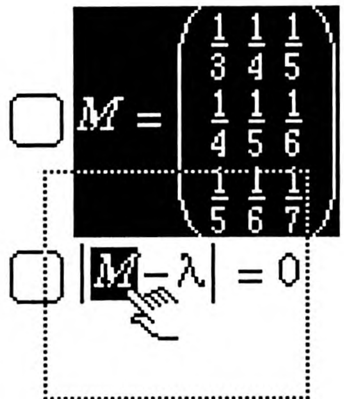
Finding Eigenvalues

You will recall that in order to find the eigenvalues, we must solve the equation $|M - \lambda I| = 0$. Make a new assumption and type **| (M-'l** esc **=0.**

$$\square\ |M - \lambda| = 0$$

Choose **Clarify** from the **Notebook** menu and declare λ as a variable.

Substitute the simplified 'M =' matrix equation into the M of this new assumption:



...so you get:

$$\triangle \left| \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} - \lambda \right| = 0$$

Expand it a couple of times:

$$\triangle -\frac{1}{4}\left(\frac{1}{4}\left[-\lambda+\frac{1}{7}\right]-\frac{1}{30}\right)+\frac{1}{5}\left(-\frac{1}{5}\left[-\lambda+\frac{1}{5}\right]+\frac{1}{24}\right)+\left(-\lambda+\frac{1}{3}\right)\left(\left[-\lambda+\frac{1}{7}\right]\left[-\lambda+\frac{1}{5}\right]-\frac{1}{36}\right)=0$$

$$\triangle -\lambda^3+\frac{71}{105}\lambda^2-\frac{317}{25200}\lambda+\frac{1}{378000}=0$$

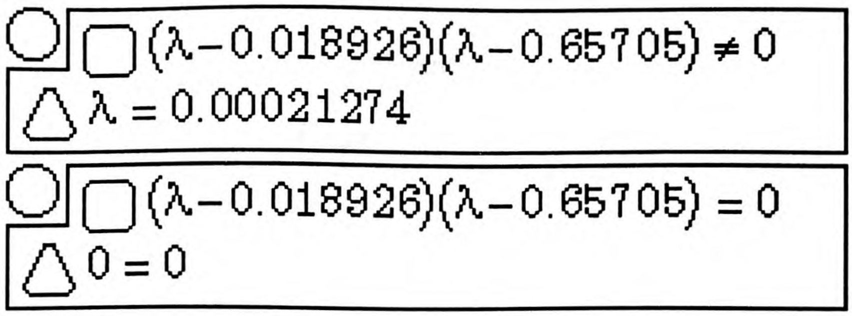
Select the polynomial (the expression on the left side of the equal sign) and Factor it. When you get a factor dialog, specify that you want a numeric answer.

$$\triangle -(\lambda-0.00021274)(\lambda-0.018926)(\lambda-0.65705)=0$$

Finding Eigenvectors

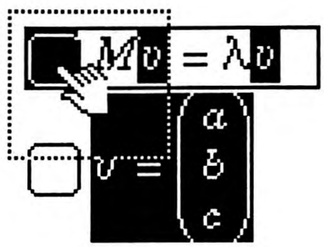
Turn on the **Auto Casing** option from the **Manipulate** menu's **Preferences** submenu. Take the first λ in the factored result and Isolate it.

These case theories appear side-by-side on your screen, but you can arrange them like this if you wish.



Create two new assumptions underneath these two theories. In the first new assumption, type **M*v='l*v**. In the second, type **v=(a;b;c)**.

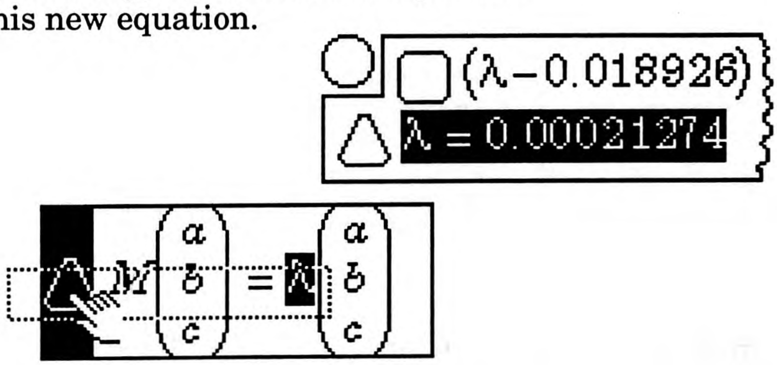
Clarify the notebook and declare v as an M-Linear op. (The names a, b, and c are declared by Theorist in the New Notebook.) Substitute the whole v equation into the $Mv = \lambda v$ equation,



...which gives you:

$$\triangle\ M\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Now, from the first case theory, Substitute the equation $\lambda = 0.00021274$ into this new equation.



The following new conclusion appears within your case theory:

$$\triangle\ M\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0.00021274\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Now take the first proposition at the beginning of your notebook,

$$\square\ M = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$$

...and Substitute it into the recent equation in the case theory:

$$\triangle\ M \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0.00021274 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

...which gives you:

$$\triangle \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0.00021274 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Select the expression to the right of the equal sign, and choose the icon from the palette's pop-up sub-palette (or choose **Move Over** from the **Manipulate** menu's **Other** submenu).

$$\triangle\ -0.00021274 \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

Expand it and you end up with a vector like this:

$$\triangle \begin{pmatrix} 0.33312a + \frac{1}{4}b + \frac{1}{5}c \\ \frac{1}{4}a + 0.19979b + \frac{1}{6}c \\ \frac{1}{5}a + \frac{1}{6}b + 0.14264c \end{pmatrix} = 0$$

These are three equations in three unknowns, all locked into one. To separate them out, select the top sum in the matrix and Isolate it.

$$\triangle\ 0.33312a + \frac{1}{4}b + \frac{1}{5}c = 0$$

Now Isolate the second row of the matrix.

At this state of the calculation, there is a catch: the three equations are not totally independent. There are really only two good equations and the third is redundant. Therefore, your 'answer' has an extra ambiguity which translates to an extra degree of freedom. You will recall that the length of an eigenvector is arbitrary.

So what we are really doing is finding *ratios* between *a, b,* and *c.* We can make our work easier by just declaring $c = 1$. Create a new assumption, type **c=1**, and select it. Shift-drag over the *c*'s in the last two conclusions you derived...

$$\triangle\ 0.33312a + \frac{1}{4}b + \frac{1}{5}c = 0$$

$$\triangle\ \frac{1}{4}a + 0.19979b + \frac{1}{6}c = 0$$

$$\square\ c = 1$$

...then choose the substitution icon from the palette's pop-up sub-palette (or choose **Substitute** from the **Manipulate** menu's **Other** submenu) to do a simultaneous substitution to eliminate *c*.

$$\triangle\ 0.33312a + \frac{1}{4}b + \frac{1}{5}c = 0$$

$$\triangle\ 0.33312a + \frac{1}{4}b + \frac{1}{5}\cdot 1 = 0$$

$$\triangle\ \frac{1}{4}a + 0.19979b + \frac{1}{6}c = 0$$

$$\triangle\ \frac{1}{4}a + 0.19979b + \frac{1}{6}\cdot 1 = 0$$

Now we have a straightforward problem with two equations in two unknowns. Take the equation $0.33312a + 1/4b + 1/5 = 0$ and solve for *a* by isolating it. Expand the answer.

$$\triangle\ a = -0.75048b - 0.60038$$

Take the resulting equation and Substitute it into *a* in the second equation.

$$\triangle\ \frac{1}{4}a + 0.19979b + \frac{1}{6}\cdot 1 = 0$$

Expand the result. Isolate *b*.

$$b = -1.3619$$

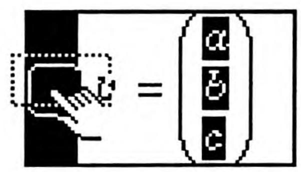
Substitute it into the *b* in the equation for *a*,

$$a = -0.75046b - 0.60038$$

...which gives us:

$$a = 0.42169$$

Now we will substitute the answers for *a, b,* and *c* into our vector. You can do them all at once by Shift-clicking on the equal sign for each answer, then ⌘-dragging or Ctrl-dragging them over the proposition icon of the destination equation.



This results in the following equation.

$$v = \begin{pmatrix} 0.42169 \\ -1.3619 \\ 1 \end{pmatrix}$$

This is your eigenvector to go along with that eigenvalue.

Make a multiple selection of the following three equations.

$$v = \begin{pmatrix} 0.42169 \\ -1.3619 \\ 1 \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$$

$$\lambda = 0.00021274$$

Substitute the selection into the $Mv = \lambda v$ equation.

$$\triangle \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 0.42169 \\ -1.3619 \\ 1 \end{pmatrix} = 0.00021274 \begin{pmatrix} 0.42169 \\ -1.3619 \\ 1 \end{pmatrix}$$

Select the right side of the equation and choose from the palette's pop-up sub-palette (or choose **Move Over** from the **Manipulate** menu's **Other** submenu). Expand the result.

$$\triangle \begin{pmatrix} -1.1723\times10^{-17} \\ 4.12\times10^{-18} \\ 3.1171\times10^{-17} \end{pmatrix} = 0$$

The numbers in scientific notation are caused by round-off error and are all effectively zero.

Exercise

By following a similar process in the other case theory, you can find the eigenvectors of the other eigenvalues.

Advanced Tutorials

Fourier Transforms

The Fourier Transform is an analysis of the frequencies present in a given input signal. The output function has a spike for every frequency that is strongly represented in the input function. Complicated input signals can lead to multiple spikes or complicated spectrums. Whereas the input signal is considered to be in the time domain (it is usually a function of time or perhaps distance), the Fourier Transform is considered to be in the frequency domain (it is usually a function of frequency or perhaps wave number).

Theorist can calculate the Fourier Transform of a table of data. The Fourier Transform and Inverse Fourier Transform commands are easily applied to a table of data. Simply select the desired table(s), then choose Fourier Transform from the Manipulate menu. (The table does not need to be named.) After the Fourier Transform is calculated (which may take a while depending upon the size of the table), a new table is created with the results of the Fourier Transform in it.

The first entry (at frequency 0) in the resulting table is the constant term, essentially an average of all values in the input table. The second entry is the strength and phase of the longest sinusoidal wave component, where one wave stretches the length of the entire input table. The second entry describes the frequency that can fit two waves into the width of the input table. The following entries represent higher frequencies.

The result of a Fourier Transform is usually a complex-valued table, because each component is not just a magnitude but also has a phase. All of this information is needed to reconstruct the original table using the Inverse Fourier Transform manipulation, which transforms a Fourier Transform back to the original function, to within round-off error. (Real-valued input functions always result in a conjugate symmetric Fourier Transform; the duplicated information is not superfluous but is needed to reconstruct the original purely real signal.)

Theorist uses a Fast Fourier Transform (FFT) algorithm to calculate Fourier transforms.

Signal with a Constant and Cosine Terms

By definition, a signal with a constant magnitude (no change in time) is composed of a real part with zero frequency. The magnitude of the constant signal is equal to the original magnitude. A signal composed of a cosine of a single frequency, when transformed, has two spikes: one at the frequency of the original frequency with a magnitude of one half the original amplitude, and another mirror-image spike.

To show this, we will use Theorist to illustrate the linearity property. In this example, we will create a table with the desired input signal, then calculate the FFT of the signal.

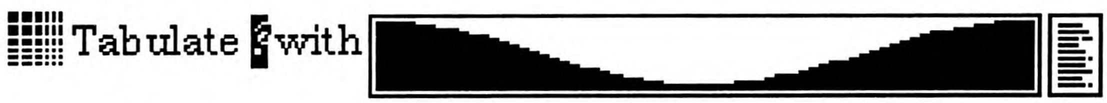
Create a new notebook and enter the following proposition.

$$\square\ 1.75+\cos(t)$$

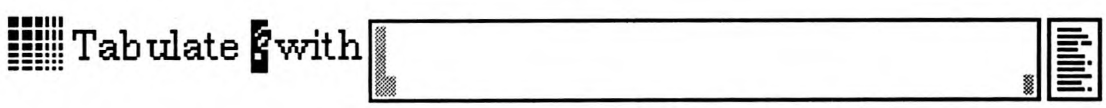
Select the expression and click the palette button (or choose **Manipulate▸Table▸Generate...**). Declare the undefined name *t* as a variable in the declaration dialog. In the New Table dialog click **OK** to accept the defaults.

When taking the Fourier Transform of a table, you should always try to make the table have a power of two number of entries (e.g., 32, 64, 1024, etc.). If the table does not have a power of two entries, Theorist pads out the table to the next highest power of two.

The resulting table looks like:



Now we are ready to take the Fourier Transform of this signal. Choose **Manipulate▸Table▸Fourier Transform**. The results of the Fourier Transform are shown below. (On a color screen, the spikes are shown in color.)



Click on the table details button to see the contents of the table.

Tabulate ?with

Save | Load | Copy | Paste

domain:	0	...	9.8438
points:	64	*inc:*	0.15625
range:	*-0.0063398*	...	*1.7682*

0	1.7682	0
0.15625	0.5035	−0.028903
0.3125	−0.0063398	0.00073052
0.46875	−0.0023349	0.00040604
0.625	−0.0012248	0.00028647
9.375	−0.0012248	−0.00028647
9.5312	−0.0023349	−0.00040604
9.6875	−0.0063398	−0.00073052
9.8438	0.5035	0.028903

Note the value of the first frequency has a magnitude of 1.768, close to the original value of 1.75. Why is the first number 1.768 instead of the original 1.75? The second, or lowest, frequency has a magnitude of 0.503, approximately half the original magnitude of the cosine term. Why is this not 1 as you might expect? Because it is split into two halves due to Fourier ambiguity.

Proper Synchronization

You may be dissatisfied that the resulting numbers seem to be so inaccurate (e.g., the 1/2 came out as 0.5035 instead of 0.5000). This is mostly because the original function had a cycle time that was a bit shorter than the length of the table. To get better results, recreate the input table, and change the high end of the domain from 6.3 to 6.283185307. (Or better yet, Calculate 2π, copy the digits, and paste them into the dialog.) Also, change the number of values in the dialog from 64 to 65. (We want both the starting and ending value to be at the same location one cycle later; the table consists of one cycle plus one extra value.) Open the table details and change the 65 to 64. Theorist automatically adjusts the ending value to be 2π minus one increment. To see this, close the table and take the Fourier Transform again, then open the details of the resulting table.

Fourier Ambiguity

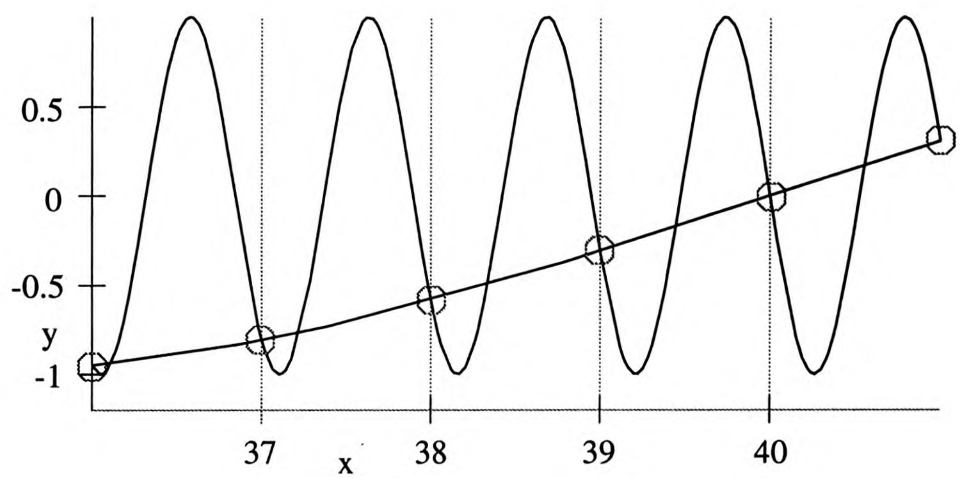
Notice there is a spike at 0 and a spike at .15915 for the main sine wave, but also another spike at the high frequency end. This is an intrinsic ambiguity inherent in Fourier Transforms. There are two ways to look at it.

One way is to say that a cosine has two components, e^{ix} and e^{-ix}, and they are the two components you see. The result of the Fourier Transform could have gone from $-n$ through n, where n is the Nyquist frequency. Then, the sine wave comes out as two spikes: one at the positive frequency and a mirror image at the same negative frequency. As it turns out, in Theorist it goes from 0 through $2n$, so the negative spike comes out at the high frequency end.

The other way of looking at it is imagining a cosine wave in a table where the domain covers the x values 0, 1, 2, ... 100. The frequency of the sine wave is 1/100. The Fourier Transform should have a spike at 0.01, which is the first entry past zero.

Now, imagine a cosine wave that had a frequency of 1 = 100/100, with one wave per value. Tabulated, it would look like there was no wave at all, that it had a constant value of 1 at frequency 0. This is an artifact of digitization that cannot be avoided unless you digitize at a greater density of points.

Now, imagine a sine wave that had a frequency of 99/100, almost as fast as 100/100 but not quite. If you tabulated the resulting wave, the tabulated numbers would look just like a wave that was 1/100 in frequency.

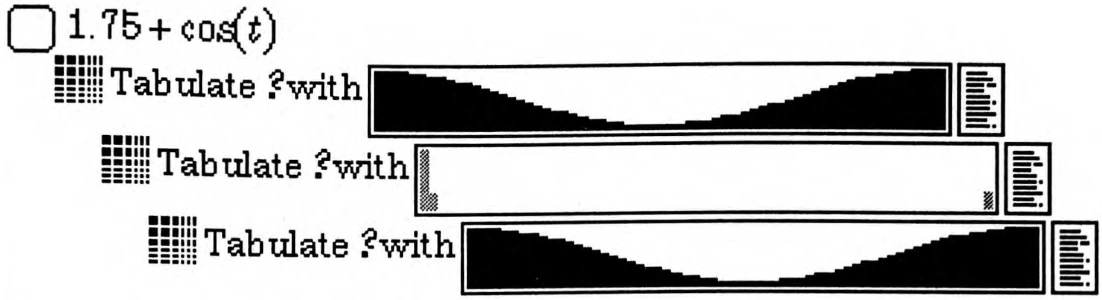


In fact, the 99/100 frequency entry in the Fourier Transform is the same entry as the –1/100 frequency entry. Theorist arranges them all starting at zero.

If you take the transform of a real function, you always get a symmetric transform like this, where the two halves are mirror images of each other. (They are equal, but complex conjugates of each other, which can lead to some interesting colors when plotted in Theorist.)

Inverse Fourier Transform

Select a table that you made with Fourier Transform then choose **Manipulate▸Table▸Inverse Fourier Transform**. Note the final table looks very similar to the original table. The result is sometimes a complex table, so on a color screen, the snapshot shows up in yellow and blue. Only round-off error precludes making it a real table and recreating the original table exactly.



This applies to both FFT and Inverse FFT.

For large tables (more than one thousand entries), the time to calculate an FFT starts to become noticeable. Even more noticeable is the time to save a notebook containing several large tables.

Exercise

Try calculating the following functions, then take their Fourier Transforms. Use default table settings.

$\sin(x^2)$	$(\sin(x))^5$	$\sin(1000(x^2+x))$
$\sin(\sin(\sin(x)))$	$(\sin(x))^{50}$	
$x - 3.15$	$(\sin(x))^{51}$	
	$(\cos(x))^{50}$	

Bessel Functions

Bessel's differential equation is given by

$$x^2 y'' + x y' + (x^2 - \nu^2)\, y = 0 \tag{1}$$

where the parameter ν is a particular number and the ′ (prime mark) denotes differentiation with respect to x. Usually ν is an integer or possibly a rational number. In Theorist, ν can be any real number. Bessel's differential equation shows up frequently in investigating a wide variety of physical problems including heat transfer, quantum mechanics, and vibrations.

The solution of (1) is known as the *Bessel function of the first kind of order* ν. If ν is not an integer, a general solution of Bessel's equation for all $x \neq 0$ is $y = A J_\nu(x) + B J_{-\nu}(x)$.

Bessel functions of the second kind of order ν are given by $Y_\nu(x)$. In some references, they are called Neumann's function and denoted by $N_\nu(x)$.

Integer values of ν are frequently denoted by *n,* which is a common case. A complete solution for Bessel's differential equation for integer ν is given by $y = A\, J_n(x) + B\, Y_n(x)$.

Modified Bessel Functions

Another differential equation similar to (1) is

$$x^2 y'' + x y' - (x^2 + \nu^2)\, y = 0 \tag{2}$$

One solution of this differential equation is the function $I_\nu(x) = i^{-\nu} J_\nu(ix)$, $i = \sqrt{-1}$, called the *modified Bessel function of the first kind of order* ν. Another solution of the differential equation (2) is the *modified Bessel function of the second kind.* This function is

$$K_\nu(x) = \frac{\pi}{2 \sin(\nu\pi)} \left[I_{-\nu}(x) - I_\nu(x) \right]$$

Using Bessel Functions

Any of the Bessel functions previously described are known to Theorist and may be used by just typing their names. For example, if we want to graph the first two orders of the Bessel function of the first kind, *J,* we would do the following.

Create a new notebook and enter the following propositions.

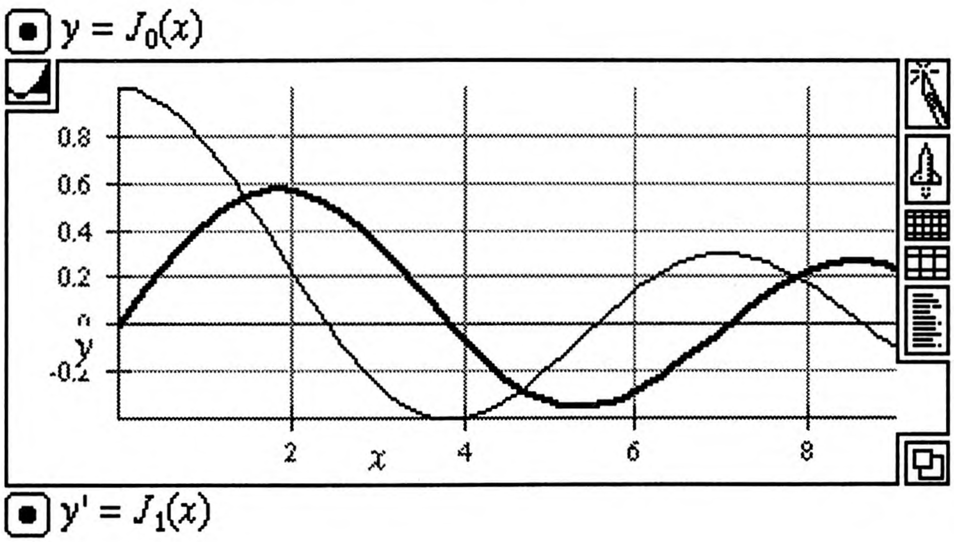
$$\square\ y = J_0(x)$$
$$\square\ y' = J_1(x)$$

Early versions of Theorist did not have predefined Bessel functions; they were in notebooks

Choose **Clarify** from the **Notebook** menu, which brings up the name declaration dialog. Click **OK** to accept the default definition for J, a predefined Bessel function. Make y' a variable.

Now we are ready to graph the Bessel functions. Select the first equation to be graphed, $y = J_0(x)$, then choose **Graph▸y = f(x)▸ Linear**.

To add the second equation $y' = J_1(x)$ to the graph, select it, then choose **Graph▸Additional▸Add Line Plot**. Accept the default graph variables. Change the domain and range.



Airy Functions

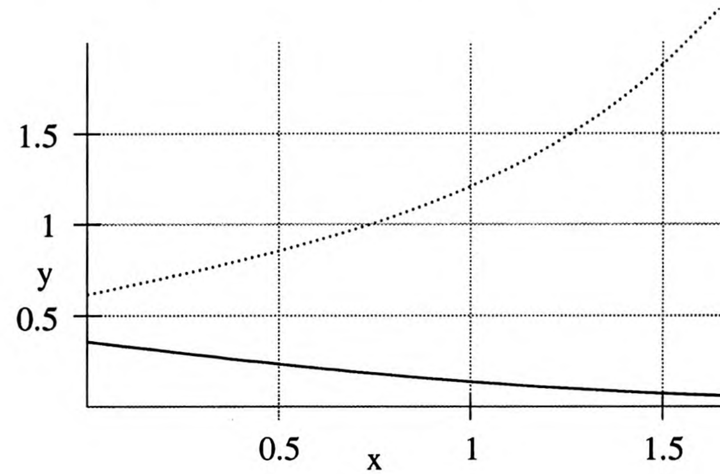
The solutions of the second-order linear differential equation

$$u'' - zu = 0 \tag{1}$$

are called *Airy functions,* which are closely related to Bessel functions. Two linearly independent solutions of (1) are

$$\mathrm{Ai}(z) = \frac{1}{\pi}\left(\frac{z}{3}\right)^{1/2} K_{1/3}\left(\frac{2z^{3/2}}{3}\right)$$
$$\mathrm{Bi}(z) = \left(\frac{z}{3}\right)^{1/2}\left[I_{-1/3}\left(\frac{2z^{3/2}}{3}\right) + I_{1/3}\left(\frac{2z^{3/2}}{3}\right)\right]$$

These are called the Airy functions of the first and second kind, respectively. Both of these functions are predefined in Theorist.



In the above graph, Ai(x) is shown with a solid line style and Bi(x) is dotted.

Transformation Rules

This chapter explores the extensions that you can create to Theorist using Transformation Rules in conjunction with wildcard variables. These rules can be designed to execute in situations that you describe, thereby giving you the power to compress several hand manipulations to a single command manipulation.

Because there is an infinite number of mathematical formulas, Theorist provides a set of programmable rules. You have full power to program these rules and apply them at your convenience. Here is an example of a typical rule:

Upon [Transform▼] transform $\cos(\mathscr{x}+\mathscr{y})$ into $\cos(\mathscr{x})\cos(\mathscr{y})-\sin(\mathscr{x})\sin(\mathscr{y})$.

The rule is easily understood: upon the Transform command (from the palette or Manipulate ▸Special submenu) it transforms the cosine of the sum of $\mathscr{x}$ and $\mathscr{y}$ according to a well-known identity.

There are two things to notice in the above example: the use of the wildcard variables $\mathscr{x}$ and $\mathscr{y}$ and the Transform pop-up menu. Wildcard variables allow you to use expressions as arguments. For example, if you have:

$$\cos(3t+2\pi)$$

...Theorist treats $3t$ as $\mathscr{x}$ and 2π as $\mathscr{y}$, when the Transform command is executed, transforming them into:

Auto Simplify is off for this example.

$$\cos(3t+2\pi) = \cos(3t)\cos(2\pi)-\sin(3t)\sin(2\pi)$$

Wildcard Variable Guidelines

It's not just a good idea, it's the law!

Be careful how you use wildcard variables; they have to correspond to the class of entities they represent:

$\mathscr{a}$ through $\mathscr{h}$ match any constants, excluding matrices
$\mathscr{i}$ through $\mathscr{n}$ match any positive integer (e.g., 3, 1042, but no names)
$\mathscr{o}$ through $\mathscr{q}$ match any expression including matrices and operators
$\mathscr{r}$ through $\mathscr{z}$ match any expression of class variable or constant

The Transform pop-up menu has two other options, Simplify and Expand. With this menu, you can choose whether the rule is activated when you do a Transform manipulation, when you Expand, or when you Simplify.

Making a Rule
Method #1

At this point you know enough preliminaries, so you can enter some rules. First, choose **Transform Rule** from the **Notebook** menu's **Insert** submenu. You get:

Upon [Transform ▾] transform ? into ?.

Type ? followed by a letter to make wildcards.

Enter the first part of the rule,

$$\int (\sin[\mathfrak{a}\,\mathfrak{x}])^2 d\mathfrak{x}$$

...into the left ?, then select the right ? and enter the second part of the rule,

$$\frac{\mathfrak{x}}{2} - \frac{\sin(2\mathfrak{a}\,\mathfrak{x})}{4\mathfrak{a}}$$

In this case, $\mathfrak{a}$ is a constant and $\mathfrak{x}$ is a variable.

Making a Rule
Method #2

There is another way of teaching Theorist this rule. Delete the previous rule (to avoid confusion later) and enter the following equation.

$$\Box \int (\sin[\mathfrak{a}\mathfrak{x}])^2 d\mathfrak{x} = \frac{\mathfrak{x}}{2} - \frac{\sin(2\mathfrak{a}\mathfrak{x})}{4\mathfrak{a}}$$

Select the equation and issue the Transform rule command again.

Upon [Transform ▾] transform $\int (\sin[\mathfrak{a}\mathfrak{x}])^2 d\mathfrak{x}$ into $\frac{\mathfrak{x}}{2} - \frac{\sin(2\mathfrak{a}\mathfrak{x})}{4\mathfrak{a}}$.

To test the rule, enter the expression:

$$\Box \int (\sin[5z])^2 dz$$

Click on the integral sign to select the expression and click the transform button from the palette (or choose **Manipulate▸Special▸Transform**).

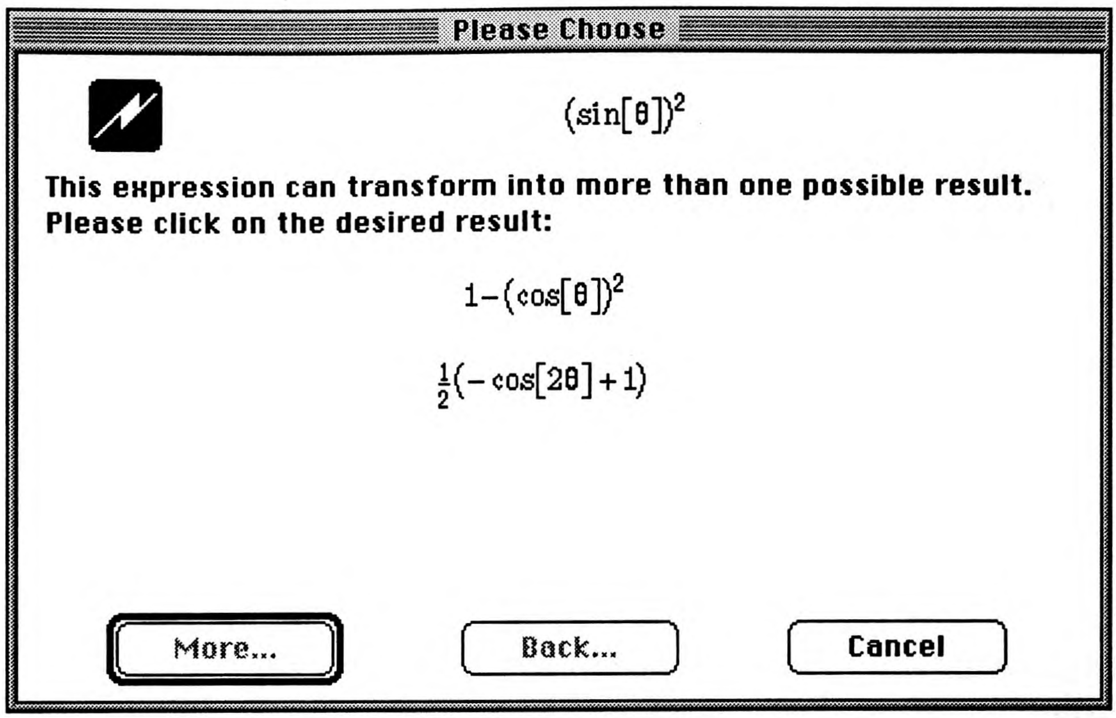
Auto Simplify is on for this example.

$$\triangle \int (\sin[5z])^2 dz = -\frac{1}{20}\sin(10z) + \frac{1}{2}z$$

Of course, the choices depend on the selection and the transformation rules in the notebook.

If you prefer, use the Transform pop-up menu to change the rule setting so the transformation is performed whenever you Simplify. Do this now. Click on the integral sign and click (or choose **Simplify** from the **Manipulate** menu), and Theorist gives you the correct result, same as the last one.

Whenever you give Theorist more than one transform for the same thing, it puts up the following dialog.



From this dialog you can choose the appropriate answer. (A previous tutorial, the Complicated Integral, makes use of this feature.)

CHEBYSHEV POLYNOMIAL GENERATION

This tutorial presents an example of Transformation rules concerning the generation of Chebyshev polynomials. The tutorial describes:

- The theory of Chebyshev polynomials
- Creating equations that describe the Chebyshev induction
- Changing these equations into Transformation Rules
- Directing these rules to execute as part of the Simplify manipulation

Theory Digression
(do not start yet)

Chebyshev (sometimes spelled Tchebychef) polynomials are a set of orthogonal polynomials used to solve differential equations. The norm defining orthogonality is:

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx$$

Given that:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

...we will derive the rest of the Chebyshev polynomials.

Now you can start.

Start with a new notebook and save it with a filename of your choice. Save it often as you do this tutorial. Consider using AutoSave.

Use [shift] [-] (under score) to enter subscripts.

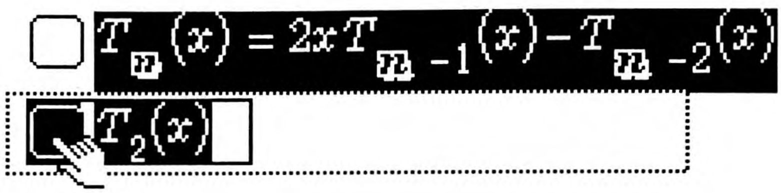
First, enter the three equations shown above. When you enter the n's, don't just type **n**—instead, type **?n** or choose $\mathfrak{n}$ from the variables palette's pop-up subpalette of wildcards. The $\mathfrak{n}$ is a wildcard variable which matches any integer. We need this because the third equation, the recurrence relation, is used later to generate others.

After you enter the equations, you have:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

Clarify the notebook and Declare T as a function.

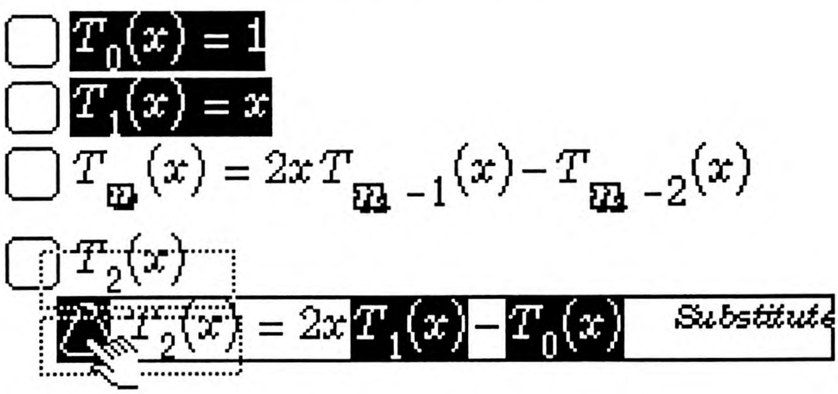
The next step is to derive the answer for $T_2(x)$. First, enter $T_2(x)$, then substitute the recurrence relation over it:



Note that the answer matches:

$$T_2(x) = 2xT_1(x) - T_0(x) \quad \textit{Substitute}$$

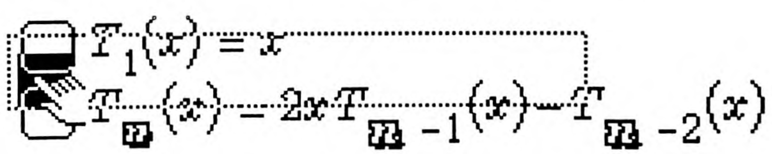
The n matches the 2. Since this gives T_2 in terms of T_1 and T_0, we need to substitute them in, too. Select the first two equations by Shift-clicking, then Substitute them over your new equation.



This yields the answer:

$$T_2(x) = 2x^2 - 1 \quad \textit{Substitute}$$

Now, move the last conclusion up under the T_1 equation.



Make a new assumption for $T_3(x)$. Substitute the recurrence relation, then substitute the equations for $T_1(x)$ and $T_2(x)$ into the resulting conclusion. Repeat by making $T_4(x)$, substituting the recurrence relation, then substituting $T_2(x)$ and $T_3(x)$.

$\square\ T_0(x) = 1$

$\square\ T_1(x) = x$

$\triangle\ T_2(x) = 2x^2 - 1$ *Substitute*

$\triangle\ T_3(x) = 2(2x^2 - 1)x - x$ *Substitute*

$\triangle\ T_4(x) = -2x^2 + 2(2[2x^2 - 1]x - x)x + 1$ *Substitute*

Expand (in place) the right sides of the $T_3(x)$ and $T_3(x)$ equations. By mathematical induction, you can generate all Chebyshev polynomials.

$\square\ T_0(x) = 1$

$\square\ T_1(x) = x$

$\triangle\ T_2(x) = 2x^2 - 1$

$\triangle\ T_3(x) = 4x^3 - 3x$

$\triangle\ T_4(x) = 8x^4 - 8x^2 + 1$

Yes, the mathematician says you can generate all Chebyshev polynomials that way. The engineer, however, says "I'm sick of doing this. Isn't there an easier way?" The answer is "Yes".

Delete your hard-earned equations, except for the original three. Select each by Shift-clicking their equal signs.

$\square\ T_0(x) = 1$

$\square\ T_1(x) = x$

$\square\ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$

Choose **Transform rule** from the **Notebook▸Insert** submenu. Theorist generates three transformation rules. Use the pop-up menus in the rules and change them from **Transform** to **Simplify**:

- [✓] Upon Simplify transform $T_0(x)$ into 1.
- [ ] $T_1(x) = x$
- [✓] Upon Simplify transform $T_1(x)$ into x.
- [ ] $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$
- [✓] Upon Simplify transform $T_n(x)$ into $2xT_{n-1}(x) - T_{n-2}(x)$.

Now, click some white space outside the propositions so nothing is selected. Make a new assumption and enter $T_7(x)$. Select and Simplify it. With the Auto Simplify preference on, Theorist repeatedly simplifies the expression, iterating the transformations up to ten times before quitting. (If you have to calculate $T_{25}(x)$, it's not much more work to simplify three times.) Expand the result, and you get the final answer.

$$\triangle\ T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x \quad \textit{Expand}$$

Important Note

Theorist looks for transformation rules from the top down, so it is important to have the rules for specific cases before those for general cases. When general cases match first, specific cases are not applied, resulting in things like $T_{-1}(x)$.

Exercise

The Legendre polynomials can be generated from the rules:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_n(x) = \frac{2n-1}{n} xP_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x)$$

Go for it.

LAPLACE TRANSFORMS

Using the Laplace transform method, the following differential equation and its associated initial conditions will be solved using Theorist.

$$y'' + 2y' + y = 1 \qquad y'(0) = 2,\ y(0) = -2$$

See the Reference Manual's Appendix C for a list of files and directories included with the Theorist disks.

Open the Laplace Transforms notebook file which you will find in the Mathematics directory included with the Theorist disks. This notebook uses Notebook Font settings with a small font size for compactness. If you like, you may increase the font size for readability. Choose **Notebook▸Windows▸Palette** so we can get a function name from the palette later. Choose **Manipulate▸Preferences▸Auto Casing** to turn it off to avoid unwanted case theories.

Clarify removes unnecessary parentheses.

Enter the above differential equation into Theorist by typing **Derivative(x)** esc **^2** esc ***y+2*Derivative(x)** esc ***y+y=1**.

$$\square \left(\frac{\partial}{\partial x}\right)^2 y + 2\frac{\partial}{\partial x} y + y = 1$$

Now we must enter the initial conditions into Theorist using a special notation convention. The nth derivative at $x = 0$ is represented as $y_{(n,0)}$. So you type return **y_0,0** esc esc **=2**, then type return **y_1,0** esc esc **=-2.**

$$\square\ y_{(0,0)} = 2$$
$$\square\ y_{(1,0)} = -2$$

Now we are ready to solve this problem. Click on the equal sign in the differential equation, then click (or choose **Select In** from the **Edit** menu). Then click on L(x) which can be found on the functions palette since it is declared in this notebook.

$$\square\ L\left(\left[\frac{\partial}{\partial x}\right]^2 y + 2\frac{\partial}{\partial x} y + y\right) = L(1)$$

Click on the equals sign in the equation, then Simplify, which performs a transformation rule as follows:

$$\triangle\ -\sum_{j=1}^{2} s^{-j+2} y_{(j-1,0)} + 2\left(-y_{[0,0]} + sY\right) + s^2Y + Y = \frac{1}{s}$$

Select the term containing the summation as shown below.

$$\triangle\ -\sum_{j=1}^{2} s^{-j+2} y_{(j-1,0)} + 2\left(-y_{[0,0]} + sY\right) + s^2Y + Y = \frac{1}{s}$$

Expand to get:

$$\triangle\ \left(-y_{[1,0]} - s y_{[0,0]}\right) + 2\left(-y_{[0,0]} + sY\right) + s^2Y + Y = \frac{1}{s}$$

Substitution is performed by holding down the ⌘ or Ctrl key and dragging the expression.

Select the initial conditions (which you typed in), then Substitute each of them into the previous expression.

$$\triangle\ \left(-y_{[1,0]} - s y_{[0,0]}\right) + 2\left(-y_{[0,0]} + sY\right) + s^2Y + Y = \frac{1}{s}$$

$$\square\ y_{(0,0)} = 2$$

$$\square\ y_{(1,0)} = -2$$

This turns the subscripted expressions into constants, as shown below.

$$\triangle\ 2(sY-2) + s^2Y + Y - 2s + 2 = \frac{1}{s}$$

Move Over the *s* on the right side to the left.

$$\triangle\ s\left(2[sY-2] + s^2Y + Y - 2s + 2\right) = 1$$

The actual display may vary slightly. You may have to manually commute terms so that all three terms with *Y* will be together as they are here.

Select the entire left side, then Expand it. This results in a polynomial in terms of *s* and *Y*.

$$\triangle\ s^3Y + 2s^2Y + sY - 2s^2 - 2s = 1$$

Select the three left-most terms that contain a *Y*,

$$\triangle\ s^3Y + 2s^2Y + sY - 2s^2 - 2s = 1$$

...then Collect.

$$\triangle \left(s^2+2s+1\right)sY-2s^2-2s=1$$

To solve this expression for *Y*, select *Y*, then Isolate.

$$\triangle Y=\frac{2s^2+2s+1}{\left(s^2+2s+1\right)s}$$

Now we will make use of a function *I*, which is defined in this notebook. Click on the equal sign in the equation, then Select In and click on I(x) from the functions palette.

$$\square I(Y)=I\left(\frac{2s^2+2s+1}{\left[s^2+2s+1\right]s}\right)$$

Now we must apply the Inverse Laplace transform to both sides of the resulting equation. For this problem, the fraction on the right side must be simplified using partial fractions. Select the fraction (inside the parentheses) on the right side, then Expand it.

$$\triangle I(Y)=I\left(\frac{1}{s}+\frac{s}{s^2+2s+1}\right)$$

These rules can be examined by exposing the collapsed definitions at the top of the notebook.

This looks like something that might match one of the Inverse Laplace transform rules in the notebook. To test this, click on the equal sign in the equation, then Select In once and Simplify.

$$\triangle y=?+1$$

Note the question mark. This indicates Theorist was unable to resolve the Inverse Laplace transform for the second term, which tells us the fraction must be reduced further still. Select the denominator in the second fraction on the right side,

$$\triangle I(Y)=I\left(\frac{1}{s}+\frac{s}{s^2+2s+1}\right)$$

...then Factor it.

$$\triangle I(Y)=I\left(\frac{1}{s}+\frac{s}{[s+1]^2}\right)$$

Now select the right fraction and Expand again.

$$\triangle\; I(Y) = I\left(\frac{1}{s} + \left[\frac{1}{s+1} - \frac{1}{\{s+1\}^2}\right]\right)$$

We are ready to try finding the Inverse Laplace transform again. Click on the equal sign in the equation, Select In, then Simplify. The result is the solution to the original differential equation.

$$\triangle\; y = e^{-x} - e^{-x}x + 1$$

Verifying the Solution

Substitute the solution into the original differential equation,

$$\left(\frac{\partial}{\partial x}\right)^2 y + 2\frac{\partial}{\partial x}y + y = 1$$

...to get the following.

$$\triangle\left(\frac{\partial}{\partial x}\right)^2\left(e^{-x} - e^{-x}x + 1\right) + 2\left(-2e^{-x} + e^{-x}x\right) + e^{-x} - e^{-x}x + 1 = 1$$

Expand the left side.

$$\triangle\; 1 = 1$$

This proves that we have found a solution to the original differential equation.

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